Exercises for Dundas' lectures. I[1]

1. If you don't know about *equivalence classes*, check it up on Wikipedia.

2. Recall what it means that a function between subsets of the real line is continuous at a point. It is continuous if it is continuous at every point. Show that the function f from $(-\infty, 0) \cup (0, \infty)$ to $\mathbb{R}$ sending x to $f(x) = 1/x$ is continuous.

3. A bijection is a function that has an inverse function, which is equivalent to saying that the function is both surjective (onto) and injective (1-1). Give an example (between subsets of the real line $\mathbb{R}$, so that you don't need to know anything fancy about topological spaces) of a continuous bijection where the inverse is not continuous (a continuous bijection where the inverse *is* continuous is called a *homeomorphism*, easily confused with the *different* algebraic notion of a homomorphism you may know of in advance).

4. Linear algebra: given a linear map ("linear transformation") F from a finite dimensional vector space V to another W, written $F\colon V \to W$. Recall what the "rank" of F is. Given bases for V and W, recall what the associated matrix is. What is the connection between the "image" of F and the "column space" of the associated matrix? Can the identity map $I_V : V \to V$ "factor" over a space of dimension less than V?

5. Linear Algebra. Let V be a vector space and $W \subseteq V$ a subspace. Define the *quotient space* V/W as the set of equivalence classes of vectors v in V modulo the relation that v is equivalent to v' if the difference $v - v'$ is in W. Let $[v]$ be the equivalence class of v. Define a vector space structure on V/W such that the quotient map $V \to V/W$ given by sending v to $[v]$ is linear. Prove that $\dim V = \dim W + \dim V/W$.

6. A **practical** exercise. Please all do it. Take a fairly large, easily orientable object (a large book, backpack or the like). Attatch some strings to your object (secure with tape or the like) and fasten the other ends of the strings to immovable objects. None of the strings should be tight. Rotate the object 360 degrees in the horizontal plane. Can you untangle the strings (we allow the strings to move on any side of the object, also below)? Does it matter how many strings you have? Rotate the object once more 360 degrees in the same direction. Same questions again.

7. Note that the "device" $\tilde{H}_0$ discussed on Monday is so that

$$\dim \tilde{H}_0 X = \#\text{ path components of } X - 1.$$

[1] The exercises (and teh links at the end of the file) will develop during the week (so you already see a few exercises for later lectures), and I'll try to incorporate issues that are discovered or discussed as they come. Please tell me if you find errors or misleading formulations.

Try to define such a device from scratch for some class of spaces you feel comfortable with. If the "minus one" is a problem: ignore it.
[Hint; check that this can be made to work: Consider the quotient of the vector space generated by all the points in X(!) by a relation where two basis elements are equivalent if the corresponding points can be joined by a (continuous) path in X.

What should be the effect of a continuous function $f\colon X \to Y$? Check "functoriality". See also Wikipedia.]

8. Run through the proof of Brower's fixed point theorem. Draw the relevant drawing. Why is the function F continuous? [you need to write down the formula for the line through two points $p \neq q$ (with $p, q \in D^2$) and solve it to find a formula the two points of unit length on this line. Notice that the formula for the point *closest to* p is continuous in (in (p, q))].

9. Show that any two continuous functions $\mathbb{R} \to \mathbb{R}$ are homotopic.

10. Do you believe that the identity map on the circle S^1 is homotopic to a constant map? Contemplate how one could determine such a thing.

11. Run through the arguments in "Somewhere the wind is still".

12. Hopefully there will be two A2-sheets present showing two green figures looking vaguely like handcuffs– imagine they are made of clay, so 3D objects. Show that one can be deformed to the other without "cutting or tearing" ("homeomorphic" as Dan discussed on Monday, but actually in a way preserving how the handcuffs lie within $\mathbb{R}^3$).

Some links

www.scientificamerican.com/article/the-strange-topology-that-is-reshaping-physics/
https://earth.nullschool.net/
https://www.youtube.com/watch?v=B4UGZEjG02s
https://en.wikipedia.org/wiki/Hairy_ball_theorem
https://en.wikipedia.org/wiki/A_Beautiful_Mind_(film)
https://en.wikipedia.org/wiki/Brouwer_fixed-point_theorem
https://en.wikipedia.org/wiki/CW_complex
https://www.youtube.com/watch?v=oCK5oGmRtxQ