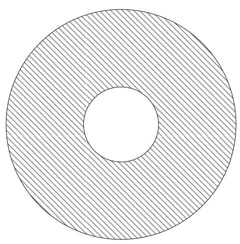
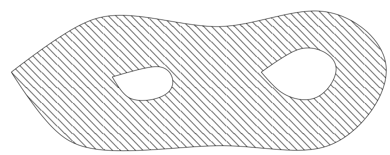
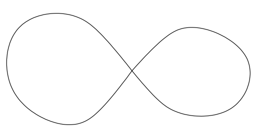
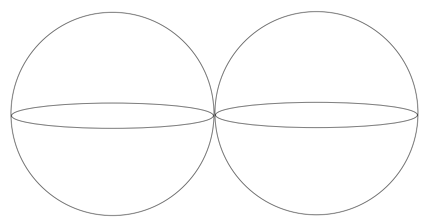
Exercises for Lecture #1 (Dugger)

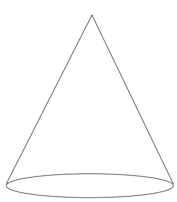
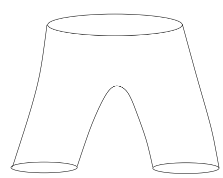
1. Classify all of the capital letters of the Latin alphabet, up to homeomorphism.
2. For each of the following spaces, find a way to regard it as a finite cell complex and compute its Euler characteristic:

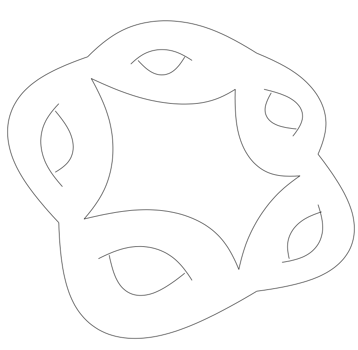
(an annulus and a double annulus)

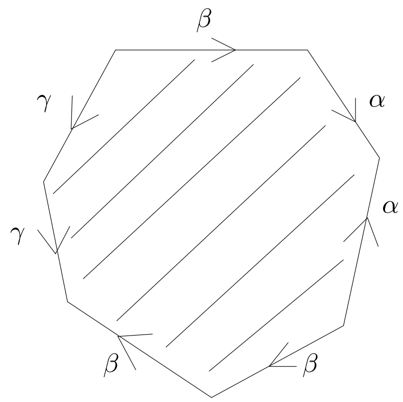
(two circles glued together at a point, and two spheres glued together at a point)

(a cone without its base, and a pair of pants)

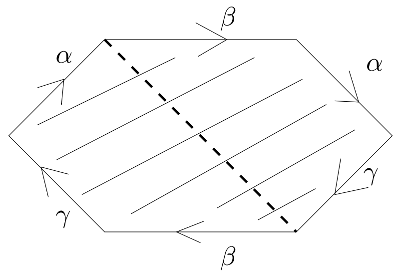


and the following quotient space:



Also, determine the Euler characteristics for $T_g = T\#T\#T\#\cdots\#T$ (g copies) and $\mathbb{R}P^2\#\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2$ (g copies). The space T_g is called the "genus g torus".

3. (a) Using cut-and-paste techniques, show that the following quotient space is homeomorphic to $\mathbb{R}P^2\#T$.



(Hint: One way is to start by cutting along the dotted line).

(b) Use cut-and-paste techniques to show that $\mathbb{R}P^2\#K \cong \mathbb{R}P^2\#T$. (Note: This might take some experimentation before you get it.)

4. If X is a topological space, the **cone on X** is the space CX obtained from $X \times I$ by collapsing $X \times \{1\}$ to a point (recall that $I = [0,1]$). The **suspension** of X is the space ΣX obtained from $X \times I$ by collapsing all of $X \times \{0\}$ to a single point, and also collapsing all of $X \times \{1\}$ to a different point. So ΣX consists of two cones CX glued together along their base.

If X is a cell complex, determine how to put related cell complex structures on $X \times I$, CX, and ΣX. Prove the formulas

(a) $\chi(CX) = 1$

(b) $\chi(X \times I) = \chi(X)$

(c) $\chi(\Sigma X) = 2 - \chi(X)$.

(Hint: To get started, maybe do all of this in the special case where X is the torus).

5. Calculate the Betti numbers of $\mathbb{R}P^2$, K, T, T_g, and $(\mathbb{R}P^2)^{\#g}$. Also calculate them for the following spaces:

(a) $S^2 \vee S^2$

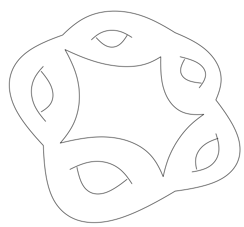
(b) $S^2 \vee T$

(c) $T \vee K$.

Guess a formula for the Betti numbers of $X \vee Y$, where X and Y are connected spaces. As a challenge, compute the Betti numbers of ΣT.

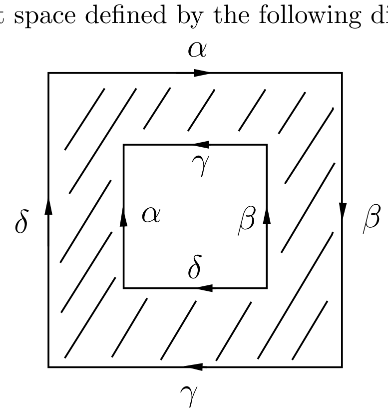
Extra problems

6. (a) Convince yourself that if X is a finite cell complex and A is a subcomplex, then $\chi(X) = \chi(A) + \chi(X/A) - 1$. Also convince yourself that if A and B are two subcomplexes such that $X = A \cup B$ and $A \cap B$ is a subcomplex, then $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.
   (b) Suppose X is a finite cell complex and A is a finite set of points in X. What is $\chi(X/A)$?
   (c) Suppose that X and Y are spaces and that we have certain points $x \in X$, and $y \in Y$. The **wedge** of X and Y is the space $X \vee Y$ obtained by gluing X and Y together by collapsing x and y into a single point. Give a formula for $\chi(X \vee Y)$ in terms of $\chi(X)$ and $\chi(Y)$. Also give a formula for $\chi(X_1 \vee X_2 \vee \cdots \vee X_n)$.
   (d) Use all the above facts to recalculate the Euler characteristics of the genus g torus T_g and of the space $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ (g factors). In other words, calculate the Euler characteristics without giving cell structures on the space.
   (e) Also, recalculate the Euler characteristic of the surface shown below (again, without giving a cell structure):



7. Show that for every $n \in \mathbb{Z}$ there exists a connected space X such that $\chi(X) = n$.

8. Let W be the quotient space defined by the following diagram:



   (a) Compute $\chi(W)$.
   (b) W is a compact 2-manifold, and it is a fact that all of these are given by the list $\{S^2, T_g, (\mathbb{R}P^2)^{\#g}\}_{g \geq 1}$. Which of these "standard models" is homeomorphic to W?
   (c) (Challenge) Find a cut-and-paste proof that W is homeomorphic to your answer in (b).

9. Let Z be the space obtained from a cube $I \times I \times I$ by making the following identifications:

   - $(x, y, 0) \sim (x, y, 1)$
   - $(x, 0, y) \sim (x, 1, y)$
   - $(0, x, y) \sim (1, x, 1 - y)$.

   Compute $\chi(Z)$. Try to compute the Betti numbers of Z.

10. [Challenge] Try to compute $\chi(S^2 \times S^2)$.