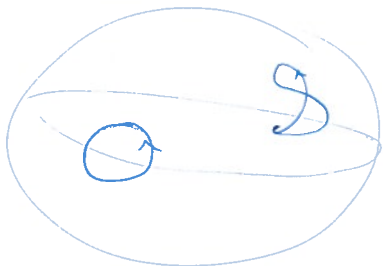
Pascal - lecture 3

Today we want to finish the argument that there are 2 curves in the sphere which are not regularly homotopic



or once and twice the equator

This is not a problem in topology because it involves derivatives. If we just wanted to look at the loops we could shrink them to a point but this is not allowed by the derivative.

What we did last time is turn this problem into a problem in topology:

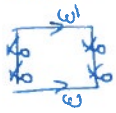
$f : [0,1] \hookrightarrow S^2$ closed immersion (C^1 and f' never 0) $\rightsquigarrow$ $\hat{f} : [0,1] \longrightarrow TS^2$, $t \longmapsto \left(f(t), \dfrac{f'(t)}{\|f'(t)\|} \right)$

this map is just continuous so this is a problem in topology.

$X = TS^2$ for instance

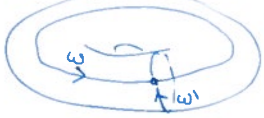
$x_0 \in X$

The point is a bit mysterious. There was not one when we started. But for some reason we need one. Based homotopy = map from rectangle



$\longrightarrow X$

Example:



ω, ω' are not homotopic

Fundamental group invented by Poincaré ~1900 (Analysis Situs)

Why is it a group? To build a new loop out of 2 ~~based loops~~ based loops you can concatenate them (go through each twice as fast in turn).

Is it clear that $(\alpha \circ \beta) \circ \gamma \overset{?}{=} \alpha \circ (\beta \circ \gamma)$. The paths are not necessarily equal but they are homotopic

The paths differ by reparametrization (by changing the clock).

Note also that if $\alpha \simeq \alpha'$ then $\alpha \cdot \beta \simeq \alpha' \cdot \beta$

The unit is ~~[illegible]~~ the constant path

The inverse a path is to follow the same path in the opposite direction.

It is an easy exercise to check this is a group.

Challenge: Find a space so that π_1 is not abelian (different from the figure eight

)

Example: 1) $\mathbb{R}^n$ $\pi_1(\mathbb{R}^n) = \{0\}$ This is because there is a very special homotopy defined on $\mathbb{R}^n$: $\mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$, $(x, u) \longmapsto ux$

H_0 is the constant map
H_1 is the identity map

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So the identity map is homotopic to the constant map. We say that such a space is contractible. The invariants we studied this week all vanish on such spaces. These spaces are like points from the point of view of Algebraic Topology.

Later we'll see that some (infinite dimensional) spaces of immersions are also contractible

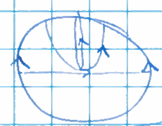
(2) $\pi_1 S^2 \cong \{1\}$ [don't write the basepoint because it can be shown that for path connected spaces groups corresponding to different points are isomorphic

Proof: Baby Van Kampen in the exercises. ☐ — If X is the union of 2 ~~sim~~ spaces with 0 fund group and connected intersection necessarily has 0 fund group.

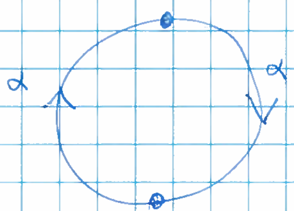
(The Big Van Kampen gives a formula for π_1 of a space when it is decomposed as a union of 2 ~~spaces~~ subspaces with connected intersection)

Note: $S^2 \setminus pt \cong \mathbb{R}^2$ so if a loop misses a point then it is contractible. There are Peano curves that pass through every point of S^2 but those can be deformed so as to miss a point.

Contractibility of the sweater on S^2:



(3) Example $\mathbb{RP}^2$:



$\alpha \cdot \alpha$ is the boundary of disc and so can be deformed to a point

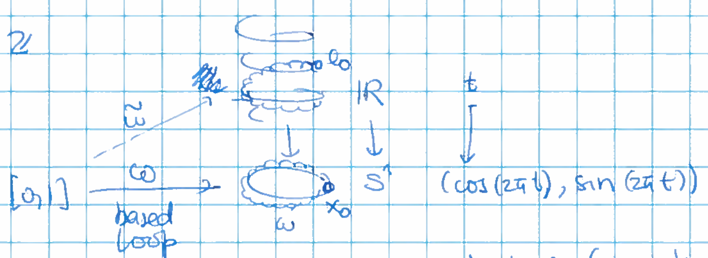
How can we see that α is not trivial?
We certainly can't do the same as before as the paths drawn above for S^2 are not loops.

We'll see that indeed $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/_2$

Challenge: ~~Write~~ Find a general formula for $\pi_1(S)$ for S given by a polygonal representation.

Solid mathematical proof of the fundamental group of the circle

$\pi_1(S^1) = ?$ Guess it is $\mathbb{Z}$



pick ℓ_0 (an integer if $x_0 = 1$)

$\tilde{\omega}$ is called a lift of ω
as the point $\omega(t)$ moves, $\tilde{\omega}(t)$ moves so that its "shadow" downstairs is always $\omega(t)$.

$\mathbb{R} \xrightarrow{p} S^1$ has the path lifting property

$$\begin{tikzcd} & \mathbb{R} = E \ni \ell_0 \text{ chosen} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\exists \tilde{\omega}"] \arrow[r, "\forall \omega"'] & S^1 = B \ni x_0 \end{tikzcd}$$

$\exists\, \tilde{\omega}$ such that $\begin{cases} p \circ \tilde{\omega} = \omega \\ \tilde{\omega}(0) = \ell_0 \end{cases}$

There are many maps in Algebraic Topology with these property.

Pedantic way to phrase this:

$$\begin{tikzcd} \{0\} \arrow[r, "b_0"] \arrow[d] & E \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\omega"] \arrow[ur, dashed] & B \end{tikzcd}$$

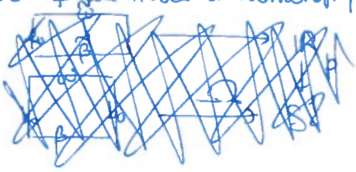
Picking $b_0 = 0$ (in $p: \mathbb{R} \to S^1$), we can define $\deg(\omega) = \tilde{\omega}(1) (= \tilde{\omega}(1) - \tilde{\omega}(0)$ for our choice of $\tilde{\omega}(0) = b_0)$

$\tilde{\omega}(1) \in p^{-1}(b_0)$ which is $\mathbb{Z}$

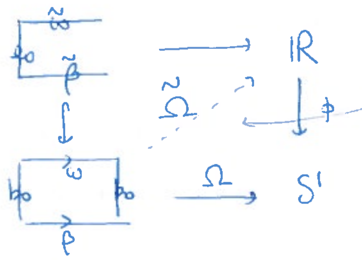
<u>Claim</u>: This trick gives $\deg: \pi_1(S^1, (1,0)) \longrightarrow \mathbb{Z}$

$[\omega] \longmapsto \tilde{\omega}(1)$

This is because if we have a homotopy of paths we can also lift the homotopy



⊛



can also be lifted so that the diagram commutes

$\tilde{\Omega}$ on the right edge lifts the constant path b_0. So it must take values in $\mathbb{Z}$ and hence be constant. Thus $\tilde{\beta}(1) = \tilde{\omega}(1)$

We'll see that ⊛ holds more generally for maps called bundles

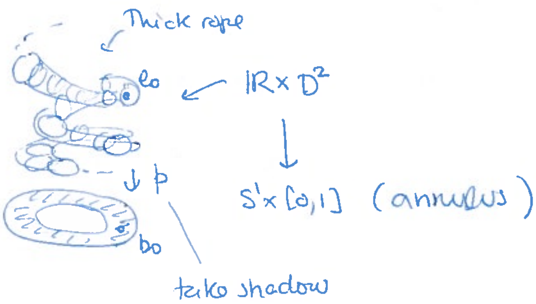
Using the fact that HLP_0 (path lifting property) we can define a map

HLP_1 (homotopy of path lifting property)

$\partial : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$ [in the case of the circle]

In general the map will take values in $\pi_0(p^{-1}(b_0))$, the set of path components of $p^{-1}(b_0)$.

<u>Example</u> (uniqueness of lift is not important)



This map p also has the path lifting property. But now we have plenty of choices (in the vertical direction)

$\tilde{\omega}$ and ω' are different lifts and have different end points but their endpoints are in the same path component of $p^{-1}(b_0)$.

The map $\partial : \pi_1(B, b_0) \longrightarrow \pi_0(E)$ is called the <u>connecting map</u>.

This is often useful to compute the fundamental group.

These path lifting properties occur when we have a <u>bundle</u> $p: E \to B$.

<u>Theorem</u>: If $p: E \to B$ is a bundle then it has HLP_n for all $n \geq 0$ (we just defined those for $n = 0, 1$ but there are analogs for all n)

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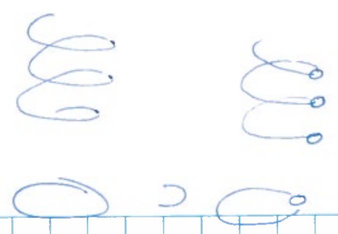
$\mathbb{R}$



$\mathbb{R}\setminus\mathbb{Z}$

$[0,1] \longrightarrow$ $\supset$ $S^1\setminus\{(1,0)\}$

here we can lift because each component of $\mathbb{R}\setminus\mathbb{Z}$ is homeomorphic to a copy of $S^1\setminus\{(1,0)\}$

$\mathbb{R}\setminus\mathbb{Z} \cong (S^1\setminus\{(1,0)\}) \times \mathbb{Z}$ so it's very easy to lift.

Back to the statement of the theorem.

Suppose $E \xrightarrow{p} B$ is a "bundle" (or more generally having HLP_n for $n=0,1$). Pick $b_0 \in B$, $e_0 \in p^{-1}(b_0)$

This is more general than being a bundle

Then we have a map $\partial : \pi_1(B,b_0) \longrightarrow \pi_0(F)$ where the fiber $F = p^{-1}(b_0)$

[To learn about bundles look at Steenrod - "The topology of fibre bundles"]

Thm: Assume E is simply connected. Then ∂ is a bijection

definition: E is simply connected (or 1-connected) if $\pi_0(E) = \{*\}$, i.e. E is path connected and $\pi_1(E) = \{1\}$

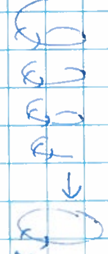
In the literature you never find such a statement because it is the "baby case" of the long exact sequence of homotopy groups of a fibration ← a map which has HLP_n for all n.

What is a bundle? It is a topological space that locally looks like a product.

Example: $\mathbb{R} \to S^1$ is a bundle with fiber $\mathbb{Z}$. It's not true that $\mathbb{R} \cong S^1 \times \mathbb{Z}$ but it's true "locally over S^1". If we take a small neighborhood N of a point in the circle

$$p^{-1}(N) \cong N \times \mathbb{Z}$$

homeomorphic by a homeomorphism h_N



N

It's a bit better than that we have moreover that

$$\begin{tikzcd} p^{-1}(N) \arrow[rr, "h_N", "\cong"'] \arrow[dr, "p"'] & & N\times\mathbb{Z} \arrow[dl, "\pi_1"] \\ & N & \end{tikzcd} \quad (*)$$

Commutes. "Locally p is like projection of a product on the first factor"

Defn: $p : E \to B$ is a bundle with fiber F if every point $b \in B$ admits a neighborhood N such that there's a homeomorphism h_N so that $(*)$ commutes

There are many examples of bundles

Note: In general $\pi_0(F)$ does not have a group structure so the theorem above is not satisfactory. However in our examples the group structure will be determined by just the cardinality of the set. Also ~~there~~ you have more structure on $\pi_0(F)$ (an action of $\pi_1 B$) which can be used to obtain information about the group

Example : $p: \mathbb{R}^2 \longrightarrow$ torus

This is a bundle

covers the torus once

$(\theta,\varphi) \longrightarrow (\cos\theta(R+\varepsilon\cos\varphi), \sin\theta(R+\varepsilon\cos\varphi), \sin\varphi)$ $R \gg 1$

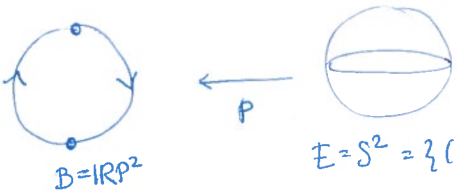
The pre-image of a patch in torus is (that patch) $\times \mathbb{Z}^2$ $p^{-1}(N) \cong N \times (\mathbb{Z}\times\mathbb{Z})$

$p \searrow \swarrow \pi_1$ N

Since $\mathbb{R}^2$ is simply connected $\partial: \pi_1(T) \longrightarrow \mathbb{Z}\times\mathbb{Z}$ is a bijection

(this doesn't quite give you the group structure)

Example : $S^2 \longrightarrow \mathbb{R}P^2$ with fibre $\mathbb{Z}/2$



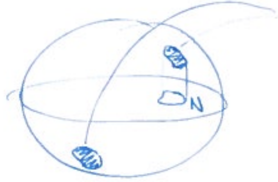
$B = \mathbb{R}P^2$ $E = S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$

Regard B as the equatorial disk bisecting the sphere.

$$p(x,y,z) = \begin{cases} \overline{(x,y,0)} & \text{if } z \geq 0 \\ \overline{(-x,-y,0)} & \text{if } z \leq 0 \end{cases}$$

$\overline{}$ denotes equivalence class in $\mathbb{R}P^2$

This seems not well defined on $z=0$ but it is by definition of $\mathbb{R}P^2$.



$p^{-1}(N)$

$\longleftarrow$ p is a bundle (check that this also works on the boundary)

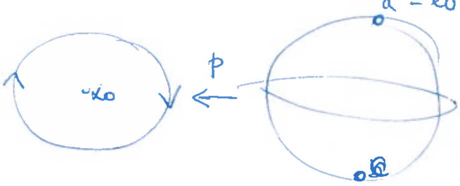
$S^2 \longrightarrow \mathbb{R}P^2$ is a bundle with fiber $F = \{a, b\} \cong \mathbb{Z}/2$. Since S^2 is simply connected (path connected and $\pi_1(S^2) = \{0\}$)

The Theorem says that $\pi_1(\mathbb{R}P^2)$ has two elements and there is only one such group.

The theorem also gives us a way of getting an explicit nonzero element in $\pi_1(\mathbb{R}P^2)$

$1 = [c_{x_0}]$

constant path at base point



$a = e_0$ $p^{-1}(x_0) = \{a, b\}$

make a choice of e_0. Say a

lift of

$\partial(\omega) = \tilde{\omega}(1)$ for a lift starting at e_0, i.e. $\tilde{\omega}(0) = e_0$

so

$\partial(\text{constant loop}) = (\text{constant loop at } e_0)(1) = e_0$

The loop which is not contractible is one so that when you lift starting at e_0 ends at e.

One can take a path from e_0 to e and project

Leave to you : a bundle has the path lifting property.

The connecting homomorphism is a bijection.

Assuming this let's try to prove the two regular homotopy classes of curves on S^2 are not regularly homotopic.

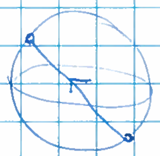
$\pi_1 TS^2 = \pi_1 SO(3) \overset{\text{claim}}{=} \mathbb{Z}/2$

Saw something related in Bjørn's lecture

Saw $H_1 = \mathbb{Z}/2$. (in fact H_1 = abelianization of π_1 for connected spaces)

$SO(3) \cong \mathbb{R}P^3 = \underbrace{D^3}_{\text{3-ball}} /_{x \sim -x}$ for x in the boundary

(**)



guess: a nontrivial loop is a diameter.

($SO(3)$ is a bundle over S^2: $TS^2 \ni (x,v)$, $\downarrow$ S^2, $\downarrow$ x; $S^1 \to SO(3) \to S^2$ but this is not what we need)

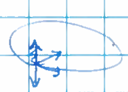
There is a bundle with fibre $\mathbb{Z}/2\mathbb{Z}$

$$\underbrace{S^3}_{\text{units in the quaternions}} \longrightarrow SO(3)$$

(This will be detailed in the exercises)

S^3 is simply connected $\Rightarrow \pi_1(SO(3)) \cong \mathbb{Z}/2$

What loop is the equator?



vertical axis is fixed and we are rotating around:

$$\omega(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) & 0 \\ \sin(2\pi t) & \cos(2\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~When we lift to the quaternions~~ so it is a path like in (**) which is a generator (which we see ~~the~~ by lifting to the quaternions).

There is a gap: we are not taking the base point into account.

There is an exercise about this subtle point. It turns out that the base point doesn't matter when π_1 is abelian (~~the~~ for such spaces ~~[illegible]~~ based and unbased homotopy classes are the same).

Note:

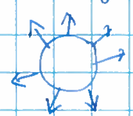
$$\{\text{regular curves in } S^2\}/_{\text{reg htpy}} \longrightarrow \{\text{based loop in } TS^2\}/_{\text{htpy}}$$

$$f: [0,1] \hookrightarrow S^2 \longmapsto \hat{f}$$

$\pi_1(TS^2)$

There is no reason why this should be surjective. This we will do tomorrow. It is the heart of Smale's theorem. In fact look

Reg. closed curves $\longrightarrow$ All loops in TS^2. This is not surjective (e.g.



is not in image because the vector is not tangent to the curve

However there is a sort of htpy inverse