Pascal - lecture 5

What have we seen so far?

- We (half) converted a geometric question to a topological question : GEOMETRY: are two regular closed curves regularly homotopic — TOPOLOGY: are the two corresponding loops homotopic

We have only seen one implication of Whitney-Graustein.
We haven't proved the converse though... This is often how algebraic topology works you get topological obstructions to geometric questions. To go back one needs more and that is where the power of Smale's theorem lies.

- Some algebraic invariants of spaces : π_0, π_1 ; Bcby Van Kampen, contractibility, connecting morphism. $\partial : \pi_1(B) \to \pi_0(F)$ for $p: E \to B$ with HLP$_K$ for $K = 0,1$; bijection if E is simply connected.

A complete (not just half) classification of immersions:

Fix a manifold M ($\mathbb{R}^2$ or S^2) for example

$$\mathcal{I} = \mathrm{Imm}([0,1], M) := \{ f : [0,1] \to M : f \text{ is an immersion} \}$$ equipped with the distance

$$d(f,g) = \sup_{t \in [0,1]} \mathrm{dist}(f(t), g(t)) + \mathrm{dist}(f'(t), g'(t))$$

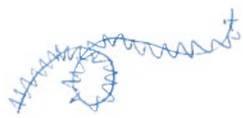
← can compare them even though they are not at the same point

two nearby immersions

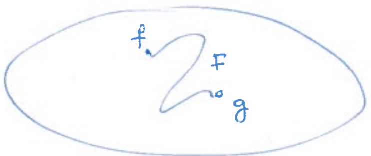
This is called the C^1-topology on the set of immersions.

An immersion is now a point in $\mathcal{I}$.

Two points f, g are regularly homotopic $\Leftrightarrow$ f, g can be connected by a path



not close because tangent vectors are not close.



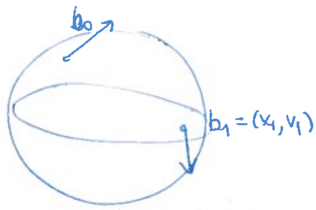
path in $\mathcal{I}$ ≡ regular homotopy

To prove the eversion of the sphere it suffices to show that the space of immersions of the sphere in $\mathbb{R}^3$ is path connected.

The space of immersions with prescribed origin and/or endpoint

Fix $b_0 = (x_0, v_0) \in T_{x_0} M = \{ (x, v) : x \in M, v \text{ is tangent at } x \text{ and } \neq 0 \}$

$M = S^2$ for instance



$$\mathcal{I}_{b_0} = \{ f \in \mathcal{I} : (f(0), f'(0)) = b_0 \}$$

$$\mathcal{I}_{b_0, b_1} = \{ f \in \mathcal{I} : (f(0), f'(0)) = b_0 , (f(1), f'(1)) = b_1 \}$$

For instance $\mathcal{I}_{b_0, b_0}$ is the space of based regular curves.

We want to prove, for example, that for $M = S^2$, $\mathcal{I}_{b_0,b_0}$ has exactly 2 path components (we already know that there are at least two). Fixing the base point does not affect much, trust me.

$\pi_0(\mathcal{I}_{b_0,b_0})$ can be accessed using the connecting homomorphism

~~This is based on 2 facts:~~

closed curves starting (and ending) at b_0.

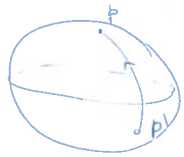
Goal : Prove that we have $\mathcal{I}_{b_0,b_0} \to \mathcal{I}_{b_0} \xrightarrow{p} T_0 M$ ← HLP for 0,1 (actually for all k)

and that $\mathcal{I}_{b_0}$ is simply connected.

We'll see that in fact

Theorem : $\mathcal{I}_{b_0}$ is contractible

Proof: To prove something is contractible we need to move all points to the base point, but one needs to be careful:

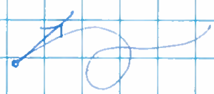


does not work, will not be continuous on South pole.

A contraction of a space is a continuous homotopy

$H : X \times [0,1] \to X$ with H_0 = constant map, $H_1 = id$.

Why is this true for immersions starting at b_0



Want to deform it to the immersion to the geodesic going in direction b_0

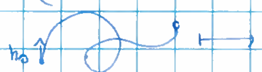
Idea: first shorten the curve and then when it is very short, project to geodesic and then rescale the start of geodesic to all of C

geodesic with beginning at b_0.

$\square$

Define $p : \mathcal{J}_{b_0} \to T_0 M$

$(f : [0,1] \hookrightarrow M) \mapsto (f(1), f'(1))$

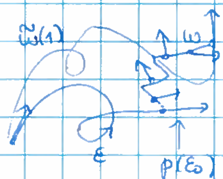


We won't prove this is a bundle but at least give the idea it has HLP_K (In the spirit of Smale's article regular curves in Riemannian manifolds)

~~Suppose~~ $\boxed{p \text{ is } HLP_0}$

$$\begin{tikzcd} \{0\} \arrow[r, "\varepsilon"] \arrow[d] & \mathcal{J}_{b_0} \arrow[d] \\ {[0,1]} \arrow[r, "\omega"] \arrow[ur, dashed, "\tilde{\omega}"] & T_0 M \end{tikzcd}$$

$\varepsilon : [0,1] \hookrightarrow M$ $\quad \varepsilon(0) = b_0$

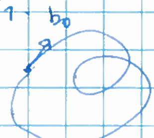


tangent vector does not follow

path lifting property means $\exists\, \tilde{\omega}$.

The details are quite technical. In the paper Smale writes an explicit formula. One has to be careful to guarantee that what one obtains is still an immersion.

Because it is an explicit formula it also allows you to obtain HLP_1.

Now we can complete the proof: The fiber $F = p^{-1}(b_0) = \mathcal{J}_{b_0, b_0} = \{$  $\}$

$\partial : \pi_0(\mathcal{J}_{b_0,b_0}) \cong \pi_1(T_b M)$ (we are using this connecting map to compute π_0 whereas before we used to compute π_1)

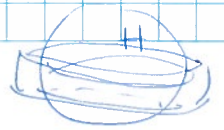
We can apply this to any M. For instance for $M = S^2$, $\pi_0 \mathcal{J}_{b_0,b_0} = \pi_1 T_0 S^2 = \pi_1 TS^2 = \pi_1 SO(3) \cong \mathbb{Z}/2$

$T_0 S^2 \cong (TS^2) \times \mathbb{R}$

There's a little work to be done to see that the curves we talked about represent the two components but this is not difficult.

Finally the eversion of the sphere (it's the same idea).

What about $Imm(S^2, \mathbb{R}^3)$? We do a standard thing in AT: cut the manifold in two pieces which are easier to understand



H = northern hemisphere plus a little bit $\cong$ Disk

A = annulus around the equator $\cong S^1 \times [-\varepsilon, \varepsilon]$

Want to show that two immersions of the sphere are homotopic.

There is no loss of generality in assuming that the 2 immersions coincide near a point. We'll assume that they coincide on the bottom. (and also on the annulus) but may be very different in the rest of the northern hemisphere

$$\begin{array}{c} \mathrm{Imm}(H, \mathbb{R}^3) \text{ frame at } b_0 \text{ fixed.} \\ \downarrow p \\ \mathrm{Imm}(A, \mathbb{R}^3) \end{array}$$

$S^2 \hookrightarrow \mathbb{R}^3$
immersion means
f is C^1
$df : \mathbb{R}^2 \to \mathbb{R}^3$ is injective at each point

The map p again is a fibration (i.e. has HLP_k for all k)

Fiber is the space of immersions with a fixed standard immersion on the bottom of the sphere.

$\pi_1(\mathrm{Imm}(A, \mathbb{R}^3)) = \pi_0(\text{fiber})$

↑

need to show this is trivial

Immersions of an annulus are just like immersions of the central circle together with a choice of transverse direction. Therefore $\pi_1(\mathrm{Imm}(A, \mathbb{R}^3)) \cong \pi_2 SO(3)$ which is known to be 0.

Perspectives : You've learned some old tools in Alg. Topology

Smale-Hirsch have managed to extend the above theory to immersions of arbitrary manifolds in another (this is from the 60s - old stuff)

Is the space non-empty? For instance you saw a glass immersion of K in $\mathbb{R}^3$

Whitney showed that M of dimension n immerses in $\mathbb{R}^{2n-1}$

$\mathbb{RP}^2 \hookrightarrow \mathbb{R}^3$
$\mathbb{RP}^3 \hookrightarrow \mathbb{R}^4$ there are obstructions to finding immersions $\rightsquigarrow$ cohomology classes $w_i \in H^i(V; \mathbb{Z}/2)$
$\mathbb{RP}^4 \hookrightarrow \mathbb{R}^6$

Smale : develop analogous theory for embeddings

Answer : Goodwillie-Klein-Weiss 2000s : embedding calculus

Embedding is ~~different~~ difficult : $\pi_0 \mathrm{Emb}(S^1, \mathbb{R}^3) = \{\text{different types of knots}\}$