Pascal lecture 2

Our goal today is to generalize Whitney-Graustein about regular closed curves in the plane to regular closed curves in any surface (or even manifold).

Recall regular curve $f:[0,1]\to\mathbb{R}^2$, f of class C^1 $f'(t)\neq(0,0)$ $\begin{matrix} f(0)=f(1) \\ f'(0)=f'(1)\end{matrix}$

Not allowed  shrinking circle because the derivative will not depend continuously on deformation parameter

Have an invariant: rotation number $\gamma(f)\in\mathbb{Z}$.

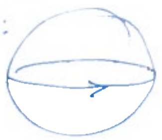
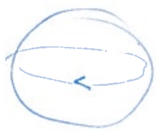
Today we'll need to introduce a very important tool in algebraic topology - the fundamental group. $\pi_1(X,x_0)$

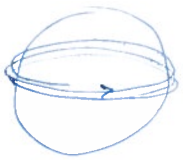
To understand this group we'll use coverings

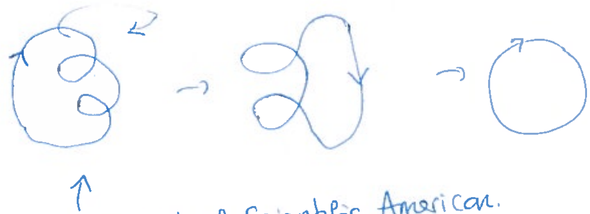
A reference for today is Smale "Regular closed curves in Riemannian manifolds".

Note: the definition of regular homotopy is not a pure topological definition.

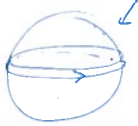
We proved half of Whitney-Graustein. We'll see the (more complicated) reverse implication in the more general context of Smale's Theorem

Example:  reg. homotopic to  (flip along y-axis  y)

 turn 3 times $\sim$ reg. homotopic to one turn



comes out of Scientific American.

last challenge:  turn 2 times $\not\sim$ just going around once

Actually we'll prove that on the sphere there are exactly two regular htpy classes of curves

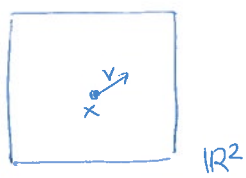
In fact there is a very simple rule to determine whether the class is one or the other: count simple double crossing points.

$f:[0,1]\xrightarrow{C^1}\mathbb{R}^2$ $(S^2,\ldots)$

$\hat{f}:[0,1]\longrightarrow\mathbb{R}^2\times S^1$

$t\longmapsto\left(f(t),\dfrac{f'(t)}{\|f'(t)\|}\right)$ [on Monday we were just looking at 2nd factor]

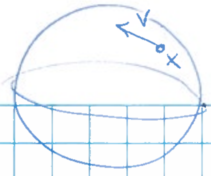
$\left.\begin{matrix} f(0)=f(1) \\ f'(0)=f'(1)\end{matrix}\right\}\Rightarrow \hat{f}(0)=\hat{f}(1)$

$\mathbb{R}^2\times S^1$ is the unit tangent bundle of the plane.  (x,v) with $x\in\mathbb{R}^2$, v of length 1, tangent to $\mathbb{R}^2$ at x

Why are   not reg. homotopic? We saw that the second components of the $\hat{f}$ maps ... circle the same

The number of times that f wrapped around S^1 was exactly $\gamma(f)$.

Let's do the same for the 2-sphere. Unit tangent bundle of S^2



$$TS^2 = \{(x,v) : x \in S^2, v \text{ tangent to } S^2 \text{ at } x, \|v\|=1\}$$

It would seem that TS^2 should be $S^2 \times S^1$

↖ the tangent vector lives on a circle.

But this is not right as we'll see in the exercises. As we move the point the circle moves and twists. It is a bit like the Möbius band (which is not homeomorphic to the cylinder).

In fact TS^2 is the space of oriented orthonormal frames in $\mathbb{R}^3$ (based at the origin)

such a frame

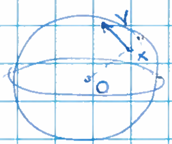
$\{e_1, e_2, e_3\}$ $\quad e_i \cdot e_j = 0$ $i \neq j$, $\quad e_i \cdot e_i = 1$ $\rightsquigarrow$ can be seen as a 3x3 matrix which is orthogonal: $A \cdot A^t = Id = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$

positively oriented means $\det A = 1$.

The set of such matrices is denoted $SO(3)$. So $TS^2 \cong SO(3)$

$\parallel$ linear rotations about $0 \in \mathbb{R}^3$

Proof:



$(x,v) \in TS^2 \iff \|x\|=1, \|v\|=1, x \perp v$

There's a unique way of building a 3rd vector which is orthogonal to those, norm 1 and forms an oriented basis

Namely we must take 3rd vector to be $x \times v$

↑ cross product.

Thus is no extra information (x,v) and $(x, v, x\times v)$ carry the same information. $\square$

$f : [0,1] \hookrightarrow S^2$ $\quad f(0)=f(1)$; $f'(0), f'(1)$ $\rightsquigarrow$ $\hat{f} : [0,1] \to TS^2 \cong SO(3)$

$t \longmapsto \left(f(t), \dfrac{f'(t)}{\|f'(t)\|}\right)$

↑ same trick as Whitney-Graustein

Want to convince you that there ~~are~~ two regular homotopy classes: this can be illustrated by the "waiter trick"...

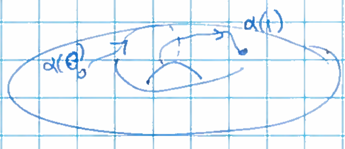
The point is $\pi_1(SO(3)) \cong \mathbb{Z}/2$ (the group with 2 elements)

We've already met another π_1 without saying so: $\pi_1 T\mathbb{R}^2 \cong \pi_1(S^1 \times \mathbb{R}^2) \cong \pi_1(S^1) \cong \mathbb{Z}$.

↑ This is why Whitney-Graustein invariant is an integer.

The fundamental group

X a space

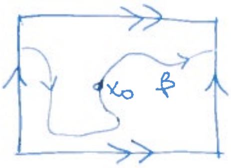


path: $\alpha : [0,1] \to X$ continuous

loop: α path with $\alpha(0) = \alpha(1)$

[don't care about derivative]

<u>Based loop</u>: $\beta : [0,1] \to X$ $\quad \beta(0)=\beta(1)$ is a fixed chosen point $x_0 \in X$.



A <u>based homotopy of loops</u> is a (continuous) map $[0,1]\times[0,1] \xrightarrow{\Omega} X$
$\qquad\qquad\qquad\qquad\qquad t \qquad u$

β on top

α at bottom

at beginning and end of each horizontal slice we are at the base point.

Can now define the fundamental group

$$\pi_1(X, x_0) = \{ \text{based homotopy classes of based loops } [\alpha] \} = \{ [\alpha] \mid \alpha \text{ is a based loop} \}$$

equivalence classes up to homotopy

<u>Example</u>: $X = S^1$ $\quad x_0 = (1,0)$

$\pi_1(S^1, x_0) = \mathbb{Z}$

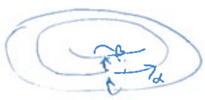
wrapping number.

We'll see a tool to actually prove this

What about the torus? $\pi_1($  $) = \mathbb{Z}\times\mathbb{Z}$

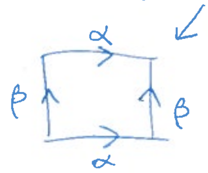
how many times we wrap in each direction.

why one



α then β same as β then α ?

It's easy to see on the square model:



Next time we'll learn how to compute π_1 and see $\pi_1 SO(3) \cong \mathbb{Z}/2$