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Bjorn lecture 5

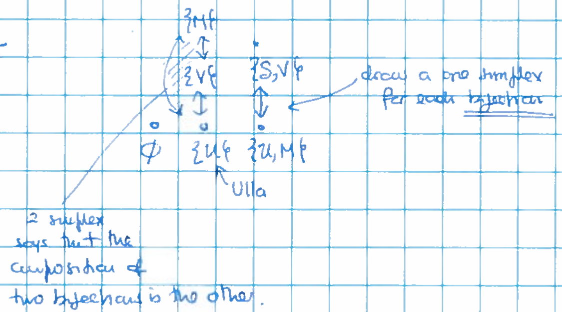
In history we have seen extensions of number system. Started with counting and added new numbers $\mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$

Today we want to go to the left hand side of the integers!

Can do operations on sets. Disjoint union ⇝ addition
cartesian product ⇝ product
↑ apply cardinality

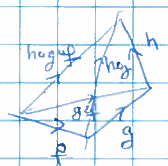
But what is a negative set? Topology is really good at subtracting things

Draw a point for a set



We are building a simplicial complex with
`set really (1-simplices can be loops)`

- 0-simplices : finite sets
- 1-simplices : bijections between them
- 2-simplices : gf g f pairs of composable bijections
- 3-simplices : triples of composable bijections

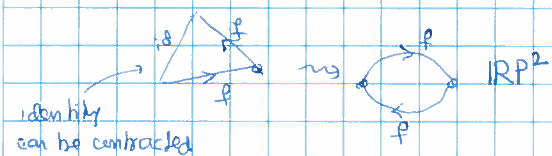


As we have more and more sheep the component corresponding to sets of cardinality one becomes a giant ∞-dimensional simplex. This is contractible so "all sheep are equal"

What about the space of 2 point sets? That is an interesting space

$\{U,V\}$ f bijection which switches them (people confuse the sheep)

If we confuse twice we get nothing $f^2 = id$

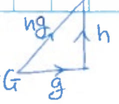


The component where U and V live contains a projective plane.

If we draw the 3-simplex carefully to $f^3 = f$ you'll see you get $\mathbb{R}P^3$ and so on.

So in fact we have $\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n$ inside that component.
$\mathbb{R}P^\infty = B\Sigma_2$ classifying space of the group with 2 elements.

G a group. BG is a space that has
- 0-simplices $*$
- 1-simplices G
- 2-simplices $G \times G$
- 3-simplices $G \times G \times G$



classifying space of G

Property of BG

$\Omega BG \simeq G$ i.e. ΩBG is an enormous fat copy of G

↑ space of all loops in BG

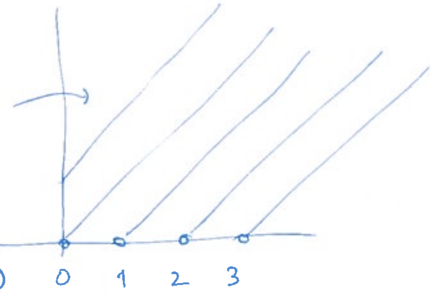
$\pi_1 BG = G$ (moreover $\Omega^k BG$ are contractible for $k>1$, i.e. all higher homotopy groups vanish)

$B\Sigma_n$ remember not just the cardinality of n-element sets but also how they interact

To build negative numbers we looked at pairs

$(4,2)$ $4-2$
$\overset{?}{\sim}$
$(3,1)$ $3-1$
$\overset{?}{\sim}$
$(2,0)$ $2-0$

new guys on the block (negative numbers)



$\mathbb{N} \rightsquigarrow \mathbb{Z}$

We can do the same thing for sets! $(m+k) \sim (n+k)$ generated the above relation.

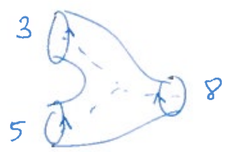
$\underset{\text{sets}}{\Sigma} \rightsquigarrow$ pairs

(X, Y) instead of taking equivalence classes, add new morphisms (1-simplexes)

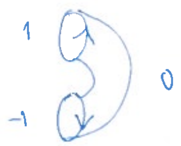
$(X,Y) \longleftrightarrow (X \amalg A, Y \amalg A)$ think of this as a pair $X-Y$

Can regard these as "negative sets". We obtain $\Sigma^{-1}\Sigma$, mimicking the natural numbers.
↑ Quillen

Another approach (string theory)



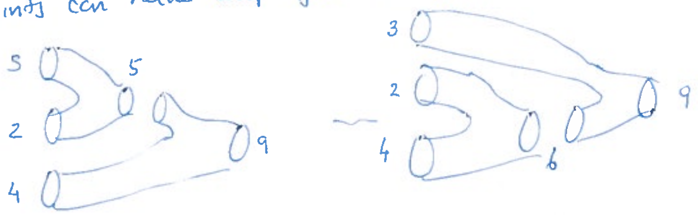
particles coliding ⇝ smooth transition from 2 to 1 strings.



particle + antiparticle. Orientation accounts for anti

Instead of points can have loops for each set.

Associativity:



~~What we are doing in taking B... and looking at ΩBN~~

$B\mathbb{N}$

also have the loop going in the negative direction and this counts as -1

$\Omega B\mathbb{N} \simeq \mathbb{Z}$

Do what we did to $\mathbb{N}$ now to Σ. "Take a loop for every set". You obtain the sphere spectrum
take the classifying space of disjoint union on finite sets

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The reason it is called the sphere spectrum $\mathbb{S}$

Consider $\Omega^n S^n$ and take $n \to \infty$. We obtain a model for

$\Omega B \Sigma$

$$\underset{\substack{\uparrow \\ \text{combinatorics}}}{\Omega B \Sigma} \sim \underset{\substack{\uparrow \\ \text{stable homotopy}}}{\Omega^\infty S^\infty}$$

There's no reason to do this with sets. Can do with vector spaces $\rightsquigarrow$ topological K-theory (more ~~[illegible]~~ intelligent cohomology theory than homology) add negative dims. Can do it with free modules over rings (e.g. abelian groups) and get algebraic K-theory.

$$\begin{array}{ccc} \overset{\text{finite sets}}{\Sigma} & \subset & \mathbb{S} \\ \downarrow \text{cardinality} & & \downarrow \\ \mathbb{N} & \subset & \mathbb{Z} \end{array}$$

This arrow forgets all symmetries

$\mathbb{S}$ has one component for each integer. The loop that confused U and H corresponds ~~to~~ in number theory to the non trivial unit $-1 \in \{-1,1\} = \mathbb{Z}^\times$

All the above is fairly old.
What's not fairly old is a part of Math that extends number theory: number theory over the sphere spectrum. Number theory concepts have analogs over the sphere spectra.

What appears now is that things which ~~[illegible]~~ are very hard to understand in number theory become conceptually much more ~~[illegible]~~ understandable when you think of them in terms of the sphere spectrum.