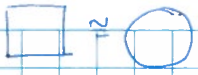
TÉCNICO LISBOA

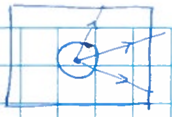
Don lecture 2 : $X \cong Y$ iff $\exists\ f: X \to Y$ continuous, $g: Y \to X$ continuous with $f \circ g = id_Y$; $g \circ f = id_X$

Have a way of matching up the points so that both it and its inverse are continuous.

Example : ①



homeomorphism given by radial projection



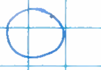
inverse is also continuous

② $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \underset{\tan}{\overset{\cong}{\longrightarrow}} \mathbb{R}$, $\xleftarrow{\arctan}$

Since distances don't matter $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is the same as any other open interval (e.g. $(0,1)$)

1-manifolds : $\mathbb{R}$ $\qquad$ S^1

not converging because the end is not there

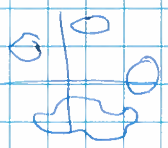


in some sense the left is infinite object and right is not. Don't mean length though. Right terminology is not compact on left and compact on right.

For the purposes of this week compact means that every sequence has a convergent subsequence.

2-manifolds : $\mathbb{R}^2$ $\qquad$ $S^2, T, \mathbb{R}P^2, K$

$\mathbb{R}^2 \setminus$ (closed set) $\qquad$ compact



there are lots of non-compact 2-dim manifolds (there are too many to think about)

Goal: Classify connected compact 2-manifolds

↑ one piece

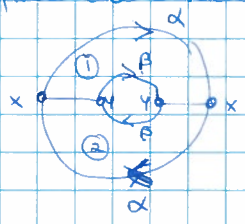
These are the fundamental objects we want to study

This was understood in early 1890s. Certainly known to Poincaré.

Last time defined Euler characteristic : $\chi(S^2)=2$; $\chi(T)=0$; $\chi(\mathbb{R}P^2)=1$; $\chi(K)=0$

Example :

$W =$



This is a 2-manifold

Let's calculate χ: Add horizontal 1-cells to get a cell decomposition

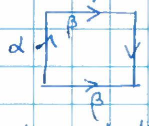
V : 2
E : 4
F : 2

$\Rightarrow \chi = 2-4+2 = 0$

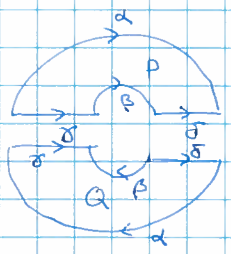
so this is not S^2 or $\mathbb{R}P^2$. It is not a torus because it is not orientable.

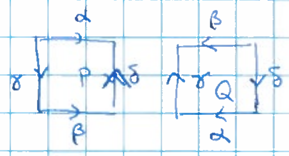
Maybe this is the Klein bottle or maybe it is something new we haven't seen yet.

Q: Is W homeomorphic to K



?

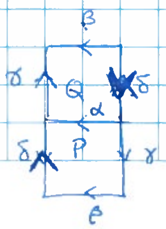




another presentation of W

We can try to chop W up so as to get to K:

Paste the α's together:



The vertical edges are being identified upside down

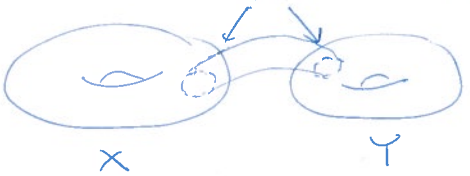
Relabeling

by

we get the picture of the Klein bottle

This is a game we can always play with these pictures. It is called cut and paste argument.

Connected sums of 2-manifolds

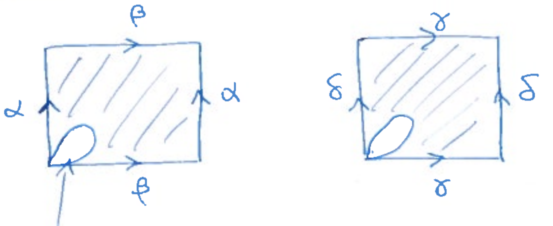


cut out small holes (leaving circles as boundary)

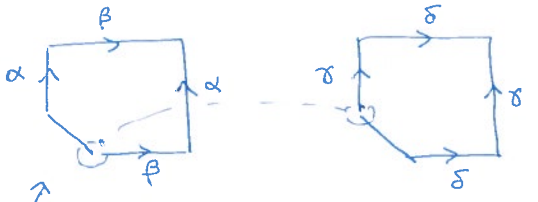
and join them with a tube (closing it back up)

The new object is called X#Y (the connected sum). Nicer picture

Here's a different way to think of this example

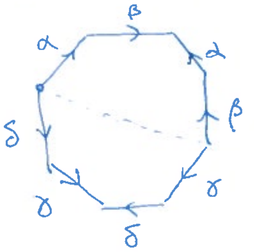


better to cut a hole here instead of the middle



new way of drawing picture on left as the beginning of α and β got identified

Now paste them together along the boundary circles to get an octagon:



all 0-cells are identified

$$\chi(T\#T) = 1 - 4 + 1 = -2.$$

this is not one of the Euler characteristics we've had so far.

Can take any polygon you like. As long as we identify the edges in pairs we obtain a new manifold. Seems like a lot...

Theorem: Every compact 2-manifold is one of the following:

$$S^2, \quad \underbrace{T\#\cdots\#T}_{g \text{ copies}} = T_g \text{ (genus g torus) or } \underbrace{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}_{g \text{ copies}} = N_g$$

non-orientable surface with g copies of $\mathbb{RP}^2$

and no two of these are homeomorphic to each other.

Sketch of a proof: For the second part use the Euler characteristic and orientability.

Harder part: take a 1000 side manifold some idiot dreamed up and prove it is one of this.

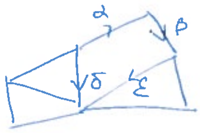
cut out a little triangle and put it on the board,

...

do it again, ...

Since the manifold is compact this will eventually end. There will be many triangles in the plane with edges identified in pairs.

Reassemble



Eventually get a polygon with identifications.

This is the first reduction. Any manifold can be written as a polygon with identifications.

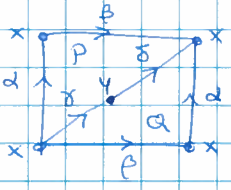
Non-trivial part of the argument: Find cut and paste rules for changing outside labels until they look like T_g and N_g. This is combinatorial. (see the exercises) □

Where is the Klein bottle on this list? Exercise: Prove $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$. We secretly did this at the beginning of todays lecture! The picture at the beginning of the lecture is $\mathbb{RP}^2 \# \mathbb{RP}^2$ (this is another exercise).

Next round of invariants after Euler characteristic: Betti #s

What I am about to do is even weirder than the Euler characteristic.

Example: Funny model for the torus



2-cells	1-cells	0-cells
P, Q	$\alpha, \beta, \gamma, \delta$	x, Y

$$0 \to \mathbb{R}^2 \longrightarrow \mathbb{R}^4 \longrightarrow \mathbb{R}^2 \to 0$$

$$\begin{array}{c|cc} & P & Q \\ \hline \alpha & 1 & -1 \\ \beta & 1 & -1 \\ \gamma & -1 & 1 \\ \delta & -1 & 1 \end{array} \qquad \begin{array}{c|cccc} & \alpha & \beta & \gamma & \delta \\ \hline x & 0 & 0 & -1 & 1 \\ Y & 0 & 0 & 1 & -1 \end{array}$$

+1 and -1 for x

γ points to Y, points from x

Get to choose the orientations of the 1-cells arbitrarily. You would think this would matter but it doesn't.

Pick an orientation for each 2-cell (clockwise for instance)

P = α, β then γ and δ backwards

It doesn't matter whether you pick clockwise or counterclockwise

list for matrices above

	$0 \to \mathbb{R}^2$	$\mathbb{R}^2 \to \mathbb{R}^4$	$\mathbb{R}^4 \to \mathbb{R}^2$	$\mathbb{R}^2 \to 0$
Rank	0	1	1	0
Nullity	0	1	3	2

numbers add up to $2 = \dim \mathbb{R}^2$

Now intertwine them as in the ⬭s above

Betti #s = Nullities - Ranks:

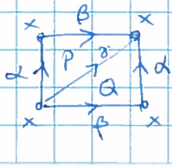
$$\left.\begin{aligned} \beta_0 &= 2-1 = 1 \\ \beta_1 &= 3-1 = 2 \\ \beta_2 &= 1-0 = 1 \end{aligned}\right\} \text{Betti \#s of torus.}$$

What did we do: Chose a cell structure (lots of choices)
Chose orientations (lots of choices)
This leads to matrices which heavily depend on the choices.

Theorem The Betti #s don't depend on the choices made.

This is amazing! Like the Euler characteristic but even weirder.

Example:



$$0 \to \underset{P,Q}{\mathbb{R}^2} \longrightarrow \underset{\alpha,\beta,\gamma}{\mathbb{R}^3} \longrightarrow \underset{x}{\mathbb{R}^1} \to 0$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

	0~~2~~	2	3	1
rank	0	1	0	0
nullity	0	1	3	1

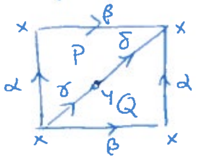
$\beta_0 = 1-0 = 1$
$\beta_1 = 3-1 = 2$
$\beta_2 = 1-0 = 1$

just like before.

Dan - Lecture 3

1st lecture : surfaces and Euler characteristic to play with
2nd lecture : Betti numbers
This lecture : will have 3 manifolds to play with.

Example from last time



$$0 \xrightarrow{\partial_3} \overset{2}{\mathbb{R}^2} \xrightarrow{\partial_2} \overset{1}{\mathbb{R}^4} \xrightarrow{\partial_1} \overset{0}{\mathbb{R}^2} \xrightarrow{\partial_0} 0$$

$$\begin{array}{c|cc} & P & Q \\ \hline \alpha & 1 & -1 \\ \beta & 1 & -1 \\ \gamma & -1 & 1 \\ \delta & -1 & 1 \end{array} \qquad \begin{array}{cccc|c} \alpha & \beta & \gamma & \delta & \\ \hline 0 & 0 & -1 & 1 & x \\ 0 & 0 & 1 & -1 & y \end{array}$$

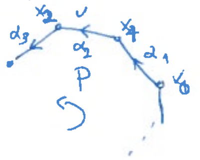
∂_i are called boundary maps

"go around the ~~cell on~~ boundary of the cell and ~~[illegible]~~ see what happens"

Key fact : $\partial_{n\cancel{+2}} \circ \partial_{n+1} = 0$

$$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & +1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This works for a geometric reason. Suppose we have a 2-cell with boundary



$$\partial_2(P) = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_{n+1}$$

$$\partial_1(\partial_2(P)) = (\cancel{x_1} - \cancel{x_0}) + (x_2 - \cancel{x_1}) + (x_3 - x_2) + \cdots + (\cancel{x_0} - x_n)$$

everything disappears.

This is an algebraic reflection of the fact that a boundary does not itself has a boundary

$\partial(\partial(\text{cell})) = \phi$ ← geometric fact

We are seeing an algebraic shadow of this fact.

$\partial_i \circ \partial_{i+1} = 0$ means $\operatorname{im} \partial_{i+1} \subset \ker \partial_i$

Can look at the quotient $\boxed{H_i(X) \underset{def}{=} \ker \partial_i / \operatorname{im} \partial_{i+1}}$ i-th homology group

$$\dim H_i(X) = \dim \ker(\partial_i) - \dim(\operatorname{im} \partial_{i+1})$$
$$= \text{nullity}(\partial_i) - \text{rank}(\partial_{i+1}) = \beta_i(X)$$

The invariant is the Betti number but we can think of it as a dimension of a vector space and $H_i(X)$ is the one that naturally comes up.

General setup : X = cell complex of dimension n
F = a field

$$C_*(X;F) = \left[0 \longrightarrow F^{r_n} \xrightarrow[\partial_n]{} F^{r_{n-1}} \xrightarrow[\partial_{n-1}]{} \cdots \xrightarrow[\partial_1]{} F^{r_0} \xrightarrow[\partial_0]{} 0 \right]$$

↑ gadget we have produced

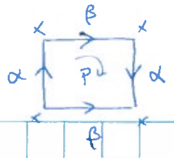
r_i = # cells ; $\partial_i \circ \partial_{i+1} = 0$ for all i

$H_i(X;F) = \ker \partial_i / \operatorname{im} \partial_{i+1}$ $\qquad \beta_i(X) = \dim H_i(X;F)$ (note these depend on the field)

↑ $\beta_i(X;F)$

The only thing you don't know is how to produce the maps ∂_n (as soon as $n>2$ we have not seen how to do this)

<u>Example</u>:



$$\begin{array}{ccccccccc} & & 2 & & 1 & & 0 & & \\ 0 & \longrightarrow & F & \longrightarrow & F^2 & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

$$\begin{array}{c|c} & \rho \\ \hline \alpha & 2 \\ \beta & 0 \end{array} \qquad \begin{array}{c|cc} & \alpha & \beta \\ \hline x & 0 & 0 \end{array}$$

<u>Homology groups of the Klein bottle</u>:

$$H_2(K;F) = \{x \in F \mid 2x=0\}/0$$

$$H_1(K;F) = F^2/\langle(2,0)\rangle$$

$$H_0(K;F) = F^1/0 = F^1$$

If $F = \mathbb{R}$ we get $H_2(K;\mathbb{R}) = 0$, $H_1(K;\mathbb{R}) = \mathbb{R}$, $H_0(K;\mathbb{R}) = \mathbb{R}$ so $\beta_i(K;\mathbb{R}) = \begin{cases} 0 & i=2 \\ 1 & i=0,1 \end{cases}$

If $F = \mathbb{F}_2$ (recall $\mathbb{F}_2 = \{0,1\}$ $1+1=0$, $1\cdot1=1$ doesn't look very useful, but it is)

In this field $2=0$ so $H_2(K;\mathbb{F}_2) = \mathbb{F}_2/0$

$H_1(K;\mathbb{F}_2) = \mathbb{F}_2^2/0$ so $\beta_i(K;\mathbb{F}_2) = \begin{cases} 1 & i=0,2 \\ 2 & i=1 \end{cases}$

$H_0(K;\mathbb{F}_2) = \mathbb{F}_2$

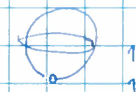
$\mathbb{C}$ and $\mathbb{Q}$ would give the same answer as $\mathbb{R}$. It is really the characteristic of the field that matters.

<u>Assignment</u>: Compute homology with $\mathbb{R}$ and $\mathbb{F}_2$ coefficients for all surfaces we've seen.

Could do this with $\mathbb{Z}$-coefficients. Would need linear algebra over $\mathbb{Z}$ which we won't talk about. Then get homology groups rather than vector spaces.

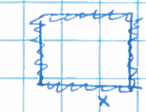
<u>3-manifolds</u>

① $S^1 =$  1 0-cell, 1 1-cell

$S^2 =$  1 0-cell, 1 2-cell

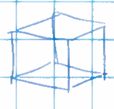
$S^3 =$ one 0-cell, no 1-cells, no 2-cells, one 3-cell

A better way to picture this: $S^2 = I \times I /$ collapse the whole boundary to a single point



$S^1 = I /$ boundary collapsed to a point

$S^3 = I \times I \times I /$ collapse the boundary to a point



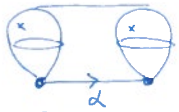
$\chi(S^3) = 1-0+0-1 = 0$

$$\begin{array}{ccccccccc} & & 3 & & 2 & & 1 & & 0 \\ 0 & \longrightarrow & F^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & F^1 \longrightarrow 0 \end{array}$$

Here there's no choice for the boundary maps. They must be zero

$$H_*(S^3;F) = \begin{cases} F & *=3 \\ 0 & *=2,1 \\ F & *=0 \end{cases}$$

② $S^2\times S^1 = \{(x,y) : x\in S^2, y\in S^1\}$ Can do this with any 2-manifold instead of S^2.



$S^2\times I$ and then must glue the 2 ends together

(this is like a torus but with the S^1 going around replaced by S^2)

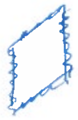
0-cell (only one in $S^2\times 0$ because the one in $S^2\times 1$ is identified with it)

1-cell   one connecting 0-cell to itself (α in picture)

2-cell   P in $S^2\times 0$ (the one on the right is identified)

3-cell   all the rest of the stuff connecting.

Another picture to keep in mind at the same time:



$I\times I$ with boundary identified to a point is S^2

cross with an interval

in every vertical slice the boundary is identified to a point. But not the same point

Finally glue the 2 end squares together.

The 0-cell corresponds to the boundaries of vertical squares at the end
the 1-cell   "   "   of the remaining vertical squares
the 2-cell   "   to the vertical squares at the ends (both identified)
the 3-cell is the interior of the cube.

What is the Euler characteristic? $\chi = 1-1+1-1=0$ (it turns out that the Euler characteristic of 3-manifolds is always zero)

Homology:

$$0 \longrightarrow F^1 \longrightarrow F^1 \longrightarrow F^1 \xrightarrow{0} F^1 \longrightarrow 0$$

$$\alpha \longmapsto 0$$

$$\uparrow$$

$P \longmapsto \partial P$ as α is a loop

∂P = go around boundary and see what you see. There are no edges so $\partial P = 0$.

$\partial(3\text{-cell}) = ?$ only care about the 2-cells on the boundary

on the horizontal boundary we are attaching to 1-dimensional cells and this doesn't count

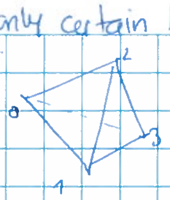
on the ends we get $\partial(3\text{-cell}) = P - P = 0$

↑ why - and not +.

Assuming this is right the boundary maps are all zero and hence $H_i(S^1\times S^2; F) = \begin{cases} F & i=0,1,2,3 \\ 0 & \text{otherwise}\end{cases}$

Bjorn also defined homology this morning and his formulas were more complicated. Ours are simpler but we are getting into those problems with signs.

His formulas are more systematic but they require using only certain kinds of cells (simplices). For those there is a ~~[illegible]~~ nice formula for the boundary which is what Bjorn used.

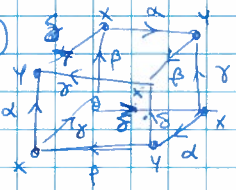


$\partial([0123]) = [123] - [023] + [013] - [012]$

Doing it our way requires fewer cells (simpler pictures) but then we run into problems in defining boundary map.

Simplicial complexes = cell complex built from simplices

Example ③



Identify opposite faces with a 90° clockwise rotation

$$0 \to F^1 \xrightarrow{\partial} F^3 \to F^4 \to F^2 \to 0$$
(degrees above: 3, 2, 1, 0)

2-cells T top square (identified w/ ~~[illegible]~~ bottom)

R right square (" with left)

F front square (" with back)

How do we take the boundary of the 3-cell ?

Claim that what's going on is more or less determined by $\partial(\partial) = 0$.

$\partial(3\text{-cell}) = T + F + R + ? + ? + ?$

Conventional ± here but the rest is determined.

for the boundaries of T along γ to cancel with F

see if we can do this as an exercise. Will solve it