Dan - Lecture 5

I want to show you a computation which will involve a fair amount of machinery.

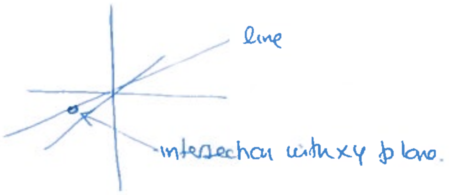
This is a problem that was raised and solved before topology existed but is really a topological problem:

Problem: Given 4 generic (randomly chosen) lines $L_1, \dots, L_4$ in $\mathbb{R}^3$. Count how many lines ℓ satisfy $\ell \cap L_i \neq \emptyset\ \forall i$

This was raised and solved by Schubert in 1880s founding the field of enumerative geometry which continues to this day.

This is a topological problem because its invariant under jiggling

$X = \{\text{lines in } \mathbb{R}^3\}$



how many ways are there of jiggling the line?

2 dims for translating point of intersection
2 dims for rotation of line

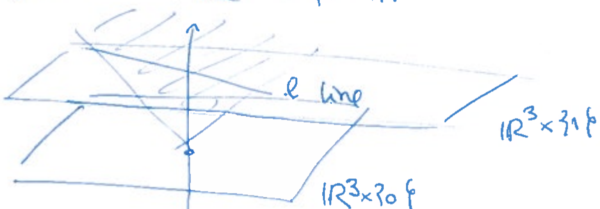
$\Rightarrow$ X is a 4-dimensional space (in fact a manifold)

lines that intersect L_1 should cut dimensions by one so get a 3 dim subspace. Doing it for $L_1, \dots, L_4$ should give a 0-dimensional space and that's just a bunch of points.

We'll spoil the fun and give the answer: 2 but we'll derive it in an interesting way.

This is not a compact manifold (can translate to ∞). What we've learned this week mostly applies to compact manifolds so we need to fix it.

$\mathbb{R}^4$



$\{\text{lines in } \mathbb{R}^3\} \hookrightarrow \{\text{planes through } 0 \text{ in } \mathbb{R}^4\}$

$\ell \longmapsto$ (join ℓ to the origin)

There are some planes which are not in the image (those contained in $\mathbb{R}^3 \times \{0\}$)

$Gr_2(\mathbb{R}^4)$ Grassmannian of 2-planes in $\mathbb{R}^4$

This is a compact manifold in $\mathbb{R}^4$.

Dodge: will Actually solve the problem with $\mathbb{R}$ replaced by $\mathbb{C}$.

The reason I am doing this is that the translation between algebra and topology works best are the complex numbers (think of roots of polynomials and how they change under jiggling). In this case we are dealing with linear equations so there is no difference and the answer is the same but we don't want to talk about this)

$Gr_2(\mathbb{C}^4)$ is an 8 dimensional compact manifold.

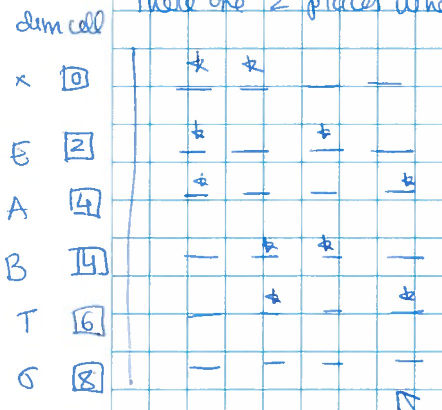
We'll find a cell decomposition although now we won't be able to draw pictures.

Consider the chain

$$0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4$$

P plane in $\mathbb{C}^4$ $\quad 0 \leq \dim P\cap\mathbb{C} \leq \dim(P\cap\mathbb{C}^2) \leq \dim(P\cap\mathbb{C}^3) \leq \dim(P\cap\mathbb{C}^4) = 2$

There are 2 places where the dimension jumps. There are six possibilities for these 2 jumps



planes where $\dim P\cap\mathbb{C} = 1$, $\dim P\cap\mathbb{C}^2 = 2$. This means $P = \mathbb{C}^2$!
so this is just one point (a 0-cell)

$P \subset \mathbb{C}^3$ and $P \supset \mathbb{C}$. Can write down a basis as

$$\begin{matrix} 1 & 0 & 0 & 0 \\ a & b & c & 0 \end{matrix}$$

Can make a go away and c can't be 0 (*) so it can be made 1

(*) unless the plane is actually in the previous cell

This decomposition is a stratification of $Gr_2(\mathbb{C}^4)$

As a consequence we obtain a canonical basis

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & b & 1 & 0 \end{matrix} \qquad b \in \mathbb{C}$$

so these form a 2-cell

For the 3rd cell a canonical basis for planes not yet accounted for are

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & b & c & 1 \end{matrix}$$

so this is a 4-cell

Next cell:

$$\begin{matrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{matrix}$$

2 degrees of freedom so a 4-cell again

can clear this entry using line above

Next cell:

$$\begin{matrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{matrix}$$

6-cell

Next cell

$$\begin{matrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{matrix}$$

8-cell (there had to be one as this is an 8 dim manifold)

To compute homology we need to know the boundary maps but here we are very lucky because as the cells are all in even dimension the boundaries must all be 0.

$$H_* Gr_2(\mathbb{C}^4) = \left\{ \begin{array}{lll} \sigma & \mathbb{Z} & *=8 \\ T & \mathbb{Z} & *=6 \\ A,B & \mathbb{Z}\oplus\mathbb{Z} & *=4 \\ E & \mathbb{Z} & *=2 \\ x & \mathbb{Z} & *=0 \end{array} \right.$$

but we have geometric descriptions of the generators

You can convince yourself that any flag can be moved to any other.

Flag in $\mathbb{C}^4$ = 4 independent dim 1 subspaces in $\mathbb{C}^4$. Moving this flag we obtain a new cell decomposition ~~[illegible]~~ and homologous cycles.

let's compute $A \cdot A = A \cdot A'$

A has been moved a little bit

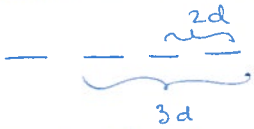
A-condition: contain first basis element

A' can for instance be those which contain the last basis element

Hence $A \cap A'$ consists of a single point.

$A \cdot B$

B = planes contained in a 3 dim subspace + intersecting a 2d subspace



← those for instance

then $A \cap B = \phi$ clearly.

$B \cdot B'$ Picking

4th and 1st coord always 0

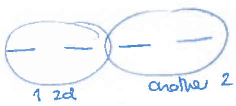
$\Rightarrow B \cap B' = \{e_2, e_3\}$

$T \cdot T \in H_{6+6-8} = H_4$ so $T \cdot T = uA + vB$

We can do this by multiplying by one more guy: $A \cdot T \cdot T = A(u \cdot A + v B) = u \cdot [x]$

$B \cdot T \cdot T = B(u \cdot A + v B) = v \cdot [x]$

$A \circ T \circ T = ?$



T = planes that intersect a 2d subspace

Pick A to be those planes that contain some line. Take that to be $e_1 + e_3$ (to make things as different as possible)

1st cond.	have	$(a\ b\ 0\ 0)$
2nd cond	have	$(0\ 0\ x\ y)$
3rd cond	contain	$(1\ 0\ 1\ 0)$

this can only be if $b = y = 0$

so intersection is $\langle e_1, e_3 \rangle \Rightarrow A \cdot T \cdot T = [x]$ and $u = 1$

$B \cdot T \circ T = ?$

B contained in 3 dimensional subspace and contains a 2d subspace. Take those to be

$P \subset \langle e_1 + e_3, e_2 + e_4 \rangle$

$P \supset \langle e_1 + e_3, e_2 + e_4, e_4 \rangle$

$\shortparallel$

$\langle e_1 + e_3, e_2, e_4 \rangle$

How many are there. A plane in the intersection contains

$a\ b\ 0\ 0$

$0\ 0\ \cancel{x}\ \cancel{y}$

3rd condition 1st and 3rd have to be the same coord

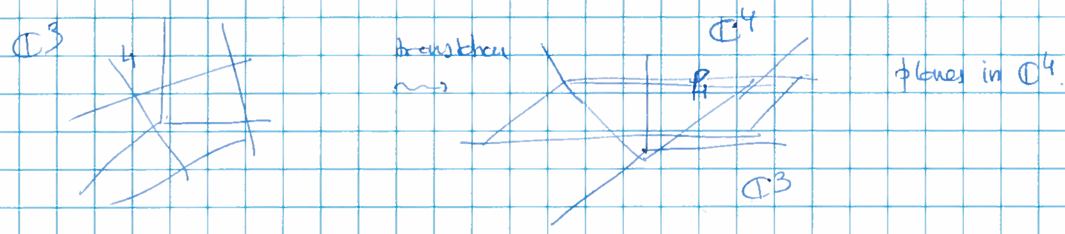
so $a = x = 0$ and $P = \langle e_2, e_4 \rangle$ so $B \cdot T \cdot T = 1 \cdot [x]$

Even if you are lost please note this is just linear algebra and we can understand this.

All the other work can be summarized as

$$T^2 = A+B$$
$$A\cdot A = B\cdot B = 1$$
$$A\cdot B = 0$$

Back to the original problem:



lines that intersect L_4 ⇝ planes in $\mathbb{C}^4$ that intersect P_1 (plane spanned by L_4 and the origin)

This is exactly captured by the cycle T. Let's call this $T_1 = T$ for a certain choice of plane.

Schubert's problem: count points in $T_1 \cap T_2 \cap T_3 \cap T_4$

$[T] = [T_1] = [T_2] = [T_3] = [T_4]$

Topological translation of Schubert's question: Compute T^4 or, backwards, computing T^4 means move T's generically (this is the generic line condition) so we can take $T_1 \cap T_2 \cap T_3 \cap T_4$.

Now we can be preschoolers:

$$T^4 = (T^2)^2 = (A+B)^2 = A^2 + 2AB + B^2 = 1 + 0 + 1 = \underset{\overset{\uparrow}{2}}{2} \;!$$