Lecture 4 - Dan

X topological space $\rightsquigarrow$ $\chi(X)$, $\beta_i(X;F)$ F field — numerical invariants

$H_*(X;F)$

Forgot to mention

Theorem: $\chi(X) = \beta_0(X;\mathbb{R}) - \beta_1(X;\mathbb{R}) + \cdots$

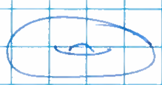
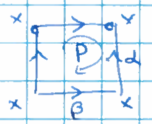
There's a trend of thinking not in terms of numerical invariants but more structured invariants. We started with $H_*(X;F)$ vector spaces and goal today is to give these more structure.

Remark: We started out studying spaces up to homeomorphism $X \cong Y$?
However the invariants you find have stronger invariance properties

$$\beta_i(X\times I) = \beta_i(X)$$
$$\chi(X\times I) = \chi(X)$$
$$H_i(X\times I) \cong H_i(X)$$

even though these spaces are not homeomorphic

This suggests that we study spaces up to the coarser relation of homotopy equivalence $\simeq$. This is what most of Algebraic Topology is about today.
There's a lesson in this: you should be ready to discard preconceived notions.

Example:  $\cong$ 

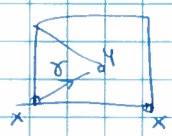
$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to 0$

$H_2(X;\mathbb{R}) = \mathbb{R}$ $\quad \mathbb{Z}$
$H_1(X;\mathbb{R}) = \mathbb{R}^2$ $\quad \mathbb{Z}^2$
$H_0(X;\mathbb{R}) = \mathbb{R}$ $\quad \mathbb{Z}$

$[x]$ basis element

Let's look at these computation a little more deeply.

What if we had chosen a more complicated ~~cell~~ cell decomposition?
Then $\partial(\gamma) = y - x$ tells you $[x]=[y]$ in $H_0(X;\mathbb{R})$



This suggests migrating to a perspective where "we consider all possible cell decompositions at once"

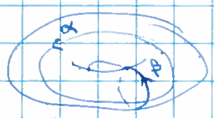
$H_0(X;\mathbb{R}) = \mathbb{R}\langle \text{points in } X\rangle \big/ \omega_0 = \omega_1 \text{ if } \exists\, \gamma : I \to X \text{ such that } \gamma(0)=\omega_0,\ \gamma(1)=\omega_1$ $= \mathbb{R}^{\#(\text{path connected components})}$

Basis for $H_1(X;\mathbb{R})$: $[\alpha], [\beta]$ in the first picture

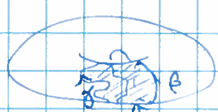
$H_i = \ker \partial_i / \operatorname{im} \partial_{i+1}$

H_1 should be (linear combinations of 1-cells that together have no boundary) / (boundaries of 2-cells)
in general

1-cycles $= \ker \partial_1$; boundaries of 2-cells $= \operatorname{im} \partial_2$

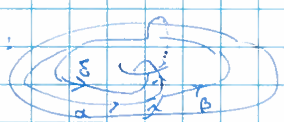


can think of many other 1-cycles, e.g.



"deformation between β and γ", i.e. a 2-dimensional thing whose boundary is $\beta - \gamma$.

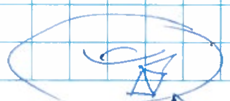
Another cycle:



$[\delta] = [\alpha] + [\beta]$

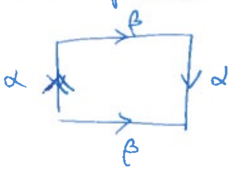
Similarly: $H_2 = (\text{linear combination of 2-cells}) / (\text{boundaries of 3-cells})$

generator of H_2 in first picture is just the unique 2-cell. But it can be seen differently



many 2-cells "tiling" the torus with boundary cancelling in pairs.

For comparison let's look at the Klein bottle



$H_0(K) = \mathbb{R}$

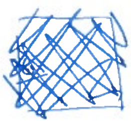
$H_1(K) = \mathbb{R}\langle \alpha, \beta \rangle / (2\alpha) = \mathbb{R}\langle \alpha \rangle$

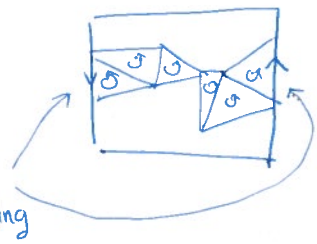
↖ boundary of 2-cell

believe that every 1-cycle can be written as a combination of α and β

we don't just know H_1, we have a geometric description of a basis element.

What about $H_2(K)$? A generator should be obtained by tiling but when we do this the ~~the~~ sides of the tilings cancel along the part of β boundary but not α. So we can't make a 2-cycle and hence $H_2(K) = 0$





not cancelling

This is the main difference between the torus and Klein bottle: orientability.

Theorem: (a) For any compact manifold, $H_*(M; \mathbb{F}_2)$ have a ring structure (a multiplication) given by intersection

(all of the H_i thought of together.)

$$u \in H_i(M; \mathbb{F}_2),\ v \in H_j(M; \mathbb{F}_2) \ > \ u \cdot v \in H_{i+j-\dim(M)}(M; \mathbb{F}_2)$$

(b) If M is orientable we get this for all coefficients.

We'll do this mostly by example.

Example: Our friend the torus

generators

$H_2 : \sigma$

$H_1 : \alpha, \beta$

$H_0 : x$

Start by

$\alpha \cdot \beta \overset{?}{=}$

Intersection is one point which is the same as x so $\boxed{\alpha \cdot \beta = x}$.

But what if we had drawn β differently?

β' now intersects α in 3 points.

But in $\mathbb{F}_2$ $3 = 1$!

What about β''? (β'' just touches α). Intersection is just two points. This is not allowed. We need the intersections to be transverse i.e. stable under "jiggling". If we alter β'' just a little bit the number of intersections goes to 1 or 3 which is the right answer.

Moral: You should only do intersection when the picture is stable under small jiggling.

What about $\alpha \cdot \alpha$?

jiggling α we get α' which does not intersect α at all so $\boxed{\alpha \cdot \alpha = 0}$

Similarly $\boxed{\beta \cdot \beta = 0}$

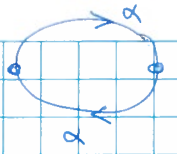
What about $\sigma \cdot \alpha$? σ is the "whole torus". It contains α ~~so~~ and this stays the same no matter how we jiggle it ~~we get~~ so $\boxed{\sigma \cdot \alpha = \alpha}$. Similarly $\boxed{\sigma \cdot \beta = \beta}$ $\boxed{\sigma \cdot x = x}$ and $\boxed{x \cdot x = 0}$

<u>Note</u>: $x \cdot x \in H_{0+0-2}(X) = H_{-2}(X)$ so we already knew this had to be 0.

↑ dim torus

It's starting to look like things time themselves is mostly 0 but this is false intuition.

<u>Example</u>:

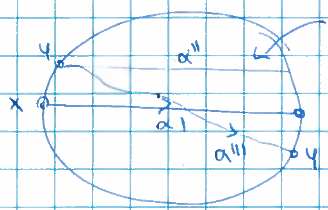


	generators
H_2	σ
H_1	α
H_0	x

When we did the Klein bottle we said the tiling would not cancel. But that was with real coefficients. But with $\mathbb{F}_2$ coefficients ~~the~~ the tiling does cancel! because $1+1=0$ so we get a σ.

This is the magic of $\mathbb{F}_2$ coefficients: it makes those orientability issues go away.

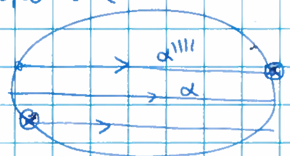
The only interesting product is $\alpha \cdot \alpha$.



bad deformation! It wouldn't be a cycle

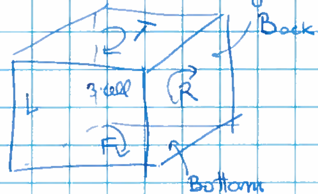
α''' is a small deformation and we would get $\alpha' \cdot \alpha''' = x$

You could consider a cycle like



but this is not a <u>small</u> jiggling. α'''' actually represents the 0 cycle. while α is ~~an~~ a non-zero class.

Let's go back to something we talked about last time:



Last time: $\partial(3\text{-cell}) \overset{?}{=}$ It should have all 2-cells and come with signs

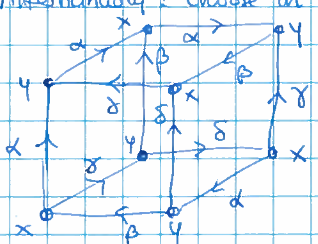
$\pm T \pm F \pm L \pm R \pm \text{Back} \pm \text{Bottom}$

How to get the signs ~~[illegible]~~
in two ways:

① $\partial^2(3\text{-cell}) = 0$ in order for the edges to cancel, the signs of T, F and R must match

Similarly ~~T,F,R~~ L, Bottom, Back ~~[illegible]~~ T must have opposite sign to

Alternatively: choose an outer normal and use the right hand rule to orient the boundaries



Chain complex

$$0 \to F \xrightarrow{0} F^3 \longrightarrow F^4 \longrightarrow F^2 \to 0$$

$$\begin{array}{c|c} & T\ \ F\ \ R \\ \hline & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \alpha\ \ \beta\ \ \gamma\ \ \delta \\ \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

(with mod 2 coefficients everything becomes +1)

$H_*(X;\mathbb{F}_2)$:
- ③ $\mathbb{F}_2$ σ ← 3-cell
- ② T+F, T+R
- ① $\alpha+\beta$, $\alpha+\gamma$
- ⓪ $\mathbb{F}_2 \cdot x$

mod 2 Betti numbers are 1, 2, 2, 1

<u>Challenge</u>: figure out the products.
- It's easy to see the 2d by 1d.
- I couldn't figure out geometrically how to do it. for intersection of

Next time we'll actually use the ring structure to do something.