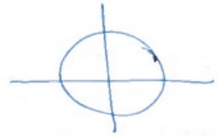
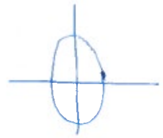
Daniel Dugger · Topology from the early days - I

Topology starts out in analytic geometry: start with perfect sphere $x^2+y^2=1$

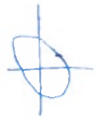


What if we jiggle (i.e. we move the 1 coefficients around)

$0.9x^2 + 1.1y^2 = 1.2$ not that different from a circle



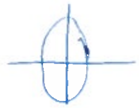
$0.8x^2 + 0.2xy + 1.1y^2 = 1.3$



in a qualitative way it is not changing that much.

If we monkey about it too much, it does change.

$x^2 + \varepsilon y^2 = 1 \underset{\varepsilon \to 0}{\rightsquigarrow} x^2 = 1$



$\varepsilon = 0$ ellipse is broken.

<u>Definition</u>: Two objects X and Y are <u>homeomorphic</u> if they can be <u>continuously deformed</u> into each other. Can stretch and shrink but not collapse.

similar shape $X \cong Y$

can be interpreted in different ways. Before used a different defn.

object means geometric object like lines. These are called topological spaces. (space: need to retrain your brain so space does not mean empty anymore)

<u>Example</u>: $A \cong D \ncong H$

how do we know? A and D have a loop (can get back to same pt without retracing ~~steps~~ your steps.

These are 1-dimensional spaces. Today want to talk about 2 dimensional spaces (26 dimensional tomorrow)

2-sphere $\cong$

and pts don't make a difference

football unified

hollow donut or bicycle tire "torus" $\cong$

<u>Definition</u>: An <u>n-dimensional manifold</u> is a space where every point looks locally like $\mathbb{R}^n$ (e.g. locally earth is flat like $\mathbb{R}^2$)

What kinds of things are not manifolds?

<u>Ex</u>: 1-manifolds

$\mathbb{R}^1$

S^1 only 1-dim manifolds   not a 1-manifold

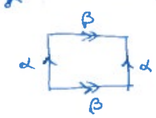
most points have small regions ~~around~~ homeomorphic to $\mathbb{R}$ but there's one bad pt where there are 2 lines crossing.

not a manifold

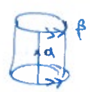
Manifolds are fund. objects like prime numbers. 2-d manifolds are understood. 3d manifolds have just been understood. Rest is still mathematics that is studied today.

How to think about 2 manifolds. Can take a flat piece of paper and glue it together

torus



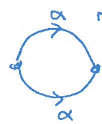
1st step



α sewed up

glue β to get a torus.

<u>Example</u>:



$\cong S^2$ $\rightsquigarrow$ get a sphere.

This process is called a quotient space $[0,1]\times[0,1]/\sim$

You can think of this as a magic transport that happens to a bug on the surface.

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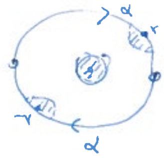
Once we have this idea we can make weird things



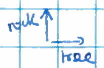
you will not be able to do this with paper (you would destroy the ...)

This is a 2-manifold

middle OK

even at the ends it still works because of the strange transport

This is called $\mathbb{R}P^2$ – the real projective plane. We'll talk about it more.

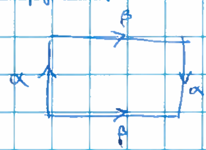


walk around S^2 or torus, when we come back everything is still the same. not on the projective plane.

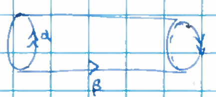
orientation has been flipped

We say S^2, torus are orientable manifolds (there's a notion of left and right that makes sense everywhere)

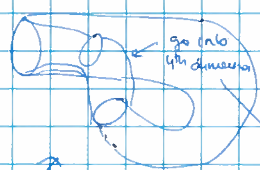
The projective plane is unorientable. Can't talk about left and right.



mix of torus and projective plane



α in the opposite direction



go into 4th dimension

would like to cross but can't do it in

let the 4th coord go up from 0 to 1/2

explain circle to a bug

move to 2nd dimension, return your steps and then come back down.

Klein bottle K (can only put it inside $\mathbb{R}^4$)

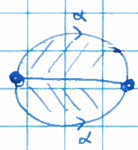
This is also not orientable

Invariants: Something we can calculate that gives information about the space.

Historically the first: Euler characteristic. Divide your space into vertices, edges and faces (these decompositions are called cell decompositions: vertices = 0-cells, edges = 1-cells, faces = 2-cells)

$$\chi = V - E + F$$

Examples.



2 vertices, 2 edges, 2 faces : $\chi = 2 - 2 + 2 = 2$

This looks silly, we could make a completely different decomposition

$V = 3$
$E = 7$
$F = 6$

$\chi = 3 - 7 + 6 = 2$

Theorem: No matter how you decompose the sphere in this way you always get 2.

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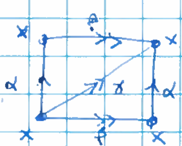
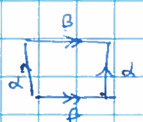


$V = 1$
$E = 3$
$F = 2$

$\chi = 1 - 3 + 2 = 0$

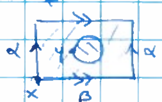
We didn't really need γ.



$V = 1$
$E = 2$
$F = 1$

$\chi = 1 - 2 + 1 = 0$

You have to be careful about one thing:



$V = 2$
$E = 3$
$F = 2$

$\chi = 2 - 3 + 2 = 1$ not right.

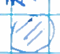
Problem: the outside is not a 2 cell. It is homeomorphic to an annulus, not a disk

So this is not a cell decomposition.

An n-cell has to be homeomorphic to $\mathbb{R}^n$.

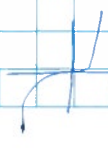
0-cell $\cong \mathbb{R}^0 = pt$

1-cell $\cong$ ⟶ $\mathbb{R}$

2-cell $\cong$  $\mathbb{R}^2$

1-cell ⟼ $\cong$ $\mathbb{R}$

given by function (tangent)



$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \underset{\arctan}{\overset{\tan}{\rightleftarrows}} \mathbb{R}$