# Plane algebraic curves - Problem set for Lectures 2-4

1. Let $F = (X^2 + Y^2)^2 - Z(Y^3 - 3X^2Y)$, $G = Y^3 - Z(Y^2 - 3X^2)$ and $P = [0 : 0 : 1]$. Use the properties of the multiplicity of intersection to compute $(F \cap G)_P$.

2. Let $F$ and $G$ be projective plane curves without common components. Denote by $m_P(F)$ and $m_P(G)$ the multiplicity of $F$ and $G$ at a point $P \in \mathbb{P}^2$.

   (a) Show that $(F \cap G)_P \geq m_P(F)m_P(G)$.

   (b) When does the strict inequality hold?

   (c) In particular, conclude that if $P$ is a non-singular point of both $F$ and $G$, and if $T_PF \neq T_PG$, then $(F \cap G)_P = 1$.

3. Let $F$ be a non-singular projective plane cubic, and $P \notin F$. Let $P_i \in F$ be the points such that $P \in T_{P_i}F$.

   (a) Show that the $P_i$'s lie on a conic.

   (b) When is this conic degenerate?

4. Let $F$ be an irreducible projective plane curve, and $P \notin F$. Suppose that $Q \in F$ is

   (a) an ordinary node; or

   (b) an ordinary cusp;

   and assume that $P$ does not lie on the tangent lines to $F$ at $Q$. Compute $(F \cap F^P)_Q$ in these cases.

**Definition.** Let $f$ be an irreducible plane curve through $P = (0, 0)$. Write

$$f = f_m + f_{m+1} + \cdots + f_d.$$

with $f_m \neq 0$ ($m = m_P(f)$).

- The linear factors of $f_m$ are called the *tangent lines to $f$ at $P$*.
- We say that $P$ is an *ordinary node* if $m = 2$; $f_2 = \ell_1\ell_2$, with $\ell_1 \neq \ell_2$; and $(f \cap \ell_i)_P = 3$ for $i = 1, 2$.
- We say that $P$ is an *ordinary cusp* if $m = 2$; $f_2 = \ell^2$; and $(f \cap \ell)_P = 3$.