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1. Show $\mathbf{Q}[\sqrt{14}]$ has Minkowski bound 3.7 and class number 1.

2. Let d be a squarefree integer with $d \equiv 1 \bmod 4$. Show

$$\mathbf{Z}\left[\frac{1+\sqrt{d}}{2}\right] = \left\{\frac{a+b\sqrt{d}}{2} : a \equiv b \bmod 2\right\}.$$

This alternate formula for elements of $\mathbf{Z}[\frac{1+\sqrt{d}}{2}]$ is sometimes more convenient to use than expressions of the form $m + n\frac{1+\sqrt{d}}{2}$.

3. In $\mathbf{Z}[\sqrt{-3}]$ check the nonunique irreducible factorization $2 \cdot 2 = (1+\sqrt{-3})(1-\sqrt{-3})$. Viewing this as an equation in $\mathbf{Z}[\frac{1+\sqrt{-3}}{2}]$, what are the prime ideal factorizations of (2), $(1+\sqrt{-3})$, and $(1-\sqrt{-3})$? Make sure your answer is consistent with unique factorization of ideals.

4. The class group of $\mathbf{Q}[\sqrt{-5}]$ has order 2, so $\mathfrak{a}^2$ is principal for every ideal in $\mathbf{Z}[\sqrt{-5}]$. Find a generator for the ideal $(3, 1+\sqrt{-5})^2$.

5. a) Show $\mathbf{Q}[\sqrt{-26}]$ has Minkowski bound 6.4 and class number 6. (Hint: Factoring $(2+\sqrt{-26})$ will give you a relation in the ideal class group between ideals of norm 2, 3, and 5.)

   b) If $\mathbf{Z}[\sqrt{-26}]$ had unique factorization (which it doesn't, by part a), show the only integral solutions of $y^2 = x^3 - 26$ are $(35, \pm 207)$, which is obviously wrong since there are also solutions $(3, \pm 1)$. (Hint: Write $x^3 = (y+\sqrt{-26})(y-\sqrt{-26})$ and show the factors on the right are relatively prime in $\mathbf{Z}[\sqrt{-26}]$.)

6. a) Show $\sqrt[3]{2} - 1$ is a unit in $\mathbf{Z}[\sqrt[3]{2}]$ with inverse $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

   b) Show $\sqrt[3]{9} - 2$ is a unit in $\mathbf{Z}[\sqrt[3]{3}]$ with inverse $4 + 3\sqrt[3]{3} + 2\sqrt[3]{9}$.

   c) Show $1 - 6\sqrt[3]{6} + 3\sqrt[3]{36}$ is a unit in $\mathbf{Z}[\sqrt[3]{6}]$ by finding an integral solution to the equation $(1 - 6\sqrt[3]{6} + 3\sqrt[3]{36})(a + b\sqrt[3]{6} + c\sqrt[3]{36}) = 1$.

7. The irreducible factorization of $x^3 - 6 \bmod 5$ is $(x-1)(x^2+x+1) \bmod 5$. Check that in $\mathbf{Z}[\sqrt[3]{6}]$ we have the ideal factorization $(5) = (5, \sqrt[3]{6} - 1)(5, \sqrt[3]{36} + \sqrt[3]{6} + 1)$. Both ideals on the right are prime, but do not check that.

8. Let p be a prime number and ζ_p be a nontrivial p-th root of unity.

   a) For $a \not\equiv 0 \bmod p$, show $(1-\zeta_p^a)/(1-\zeta_p) \in \mathbf{Z}[\zeta_p]^\times$. (Hint: First show the ratio is in $\mathbf{Z}[\zeta_p]$ and then show the reciprocal ratio is in $\mathbf{Z}[\zeta_p]$ by writing ζ_p as a power of ζ_p^a.)

   b) From the polynomial factorization $X^p - 1 = (X-1)\prod_{a=1}^{p-1}(X - \zeta_p^a)$ show $(p) = (1-\zeta_p)^{p-1}$ as ideals in $\mathbf{Z}[\zeta_p]$. (Hint: Divide by $X-1$ in the polynomial factorization, set $X = 1$ on both sides, and then use part a.)