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Finding the units in $\mathbf{Z}[\sqrt{11}]$ is the same as finding integral solutions of $x^2 - 11y^2 = \pm 1$. The simplest solution of $x^2 - 11y^2 = 1$ is $(x, y) = (10, 3)$. The theorem below says this solution leads to all others.

Theorem 1. *Every unit in* $\mathbf{Z}[\sqrt{11}]$ *has the form* $\pm(10 + 3\sqrt{11})^k$ *for some integer* k*.*

Proof. For any unit u in $\mathbf{Z}[\sqrt{11}]$, the four units $u, -u, 1/u, -1/u$ are each in one of the intervals $[1, \infty)$, $(0, 1]$, $[-1, 0)$, or $(-\infty, -1]$, so one of them is greater than or equal to 1. If one of these four units has the form $\pm(10 + 3\sqrt{11})^k$ then they all do, so without loss of generality we can assume $u \geq 1$. Our goal is to show in this case that $u = (10 + 3\sqrt{11})^k$ for some $k \geq 0$.

It is geometrically easy to see that we can place u between two powers of $10 + 3\sqrt{11}$: the numbers $(10 + 3\sqrt{11})^k$ for $k \geq 0$ start at 1 and tend monotonically to ∞, so there is some $k \geq 0$ such that

$$(10 + 3\sqrt{11})^k \leq u < (10 + 3\sqrt{11})^{k+1}. \tag{1}$$

Note we have $\leq$ on the left and $<$ on the right. The inverse of $10 + 3\sqrt{11}$ is $10 - 3\sqrt{11}$ (any unit with norm 1 has its inverse equal to its conjugate), so multiplying through (1) by $(10 - 3\sqrt{11})^k$ gives us

$$1 \leq u(10 - 3\sqrt{11})^k < 10 + 3\sqrt{11}. \tag{2}$$

The number in the middle of (2) is a unit (it is a product of two units, u and $(10 - 3\sqrt{11})^k$), so we are reduced to proving the following.

Claim: There is no unit in $\mathbf{Z}[\sqrt{11}]$ lying strictly between 1 and $10 + 3\sqrt{11}$.

If the claim is true, then from (2) we get $u(10-3\sqrt{11})^k = 1$, so $u = (10-3\sqrt{11})^{-k} = (10+3\sqrt{11})^k$ and we'd be done.

To prove the claim, we suppose there is a unit u satisfying $1 < u < 10 + 3\sqrt{11}$ and we will get a contradiction. (Note, of course, that u now has a different meaning than it did before.) Write $u = a + b\sqrt{11}$ for some unknown integers a and b.

In principle the norm of u is 1 or -1: $a^2 - 11b^2 = \pm 1$. However, there is no unit in $\mathbf{Z}[\sqrt{11}]$ with norm -1 since the equation $a^2 - 11b^2 = -1$ implies $a^2 = -1 \bmod 11$ and $-1 \bmod 11$ is not a square (the squares mod 11 are 0, 1, 3, 4, 5, and 9).

Since u has norm 1, $u^{-1} = \overline{u} = a - b\sqrt{11}$. In the inequalities

$$1 < a + b\sqrt{11} < 10 + 3\sqrt{11}. \tag{3}$$

reciprocate all the terms and we get

$$10 - 3\sqrt{11} < a - b\sqrt{11} < 1. \tag{4}$$

Adding (3) and (4) together, we get

$$11 - 3\sqrt{11} < 2a < 11 + 3\sqrt{11}. \tag{5}$$

In particular, $a > \frac{1}{2}(11 - 3\sqrt{11}) = \frac{1}{2}(\sqrt{11} - 3)\sqrt{11} > 0$, so $a \geq 1$ (remember a is an integer). More precisely, using decimal approximations for the first and third terms in (5) we get $.52 < a < 10.48$, so $1 \leq a \leq 10$.

To get inequalities on b, negate the terms in (4):

$$-1 < b\sqrt{11} - a < 3\sqrt{11} - 10.$$

Adding this to (3), we get $0 < 2b\sqrt{11} < 6\sqrt{11}$, so $1 \leq b < 3$.

Because $a^2 - 11b^2 = 1$, $a^2 = 11b^2 + 1$. If $a = 1$ then $b = 0$ and $u = a + b\sqrt{11} = 1$, but $1 < u$. If $a = 2, \ldots, 9$ then there is no integer b that fits $a^2 = 11b^2 + 1$. If $a = 10$ then $b^2 = 9$, so $b = 3$ since $b \geq 0$. However we saw that $b < 3$, so we have a contradiction.

This concludes the proof of the claim. ■

The argument used here can be adapted to describe all the units in $\mathbf{Z}[\sqrt{d}]$ for specific choices of d: a particular unit greater than 1 generates all the units (up to multiplication by a sign). There could be units with norm -1 (like $1+\sqrt{2}$ in $\mathbf{Z}[\sqrt{2}]$), so you would need to consider units with norm 1 and norm -1 in the analogue of the Claim from the proof above.