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These exercises vary in difficulty and importance. Try the ones that interest you. As in the lectures, p is always a prime number.

1. Suppose $p > 2$.

   a) Show that $\mathbf{Z}_p^\times \cong (\mathbf{Z}/p)^\times \times (1 + p\mathbf{Z}_p)$. Hint: use an earlier exercise.

   b) Show that every $x \in 1 + p\mathbf{Z}_p$ is a square in $\mathbf{Z}_p^\times$. Hint: Hensel's Lemma.

   c). Show that $\mathbf{Q}_p^\times/(\mathbf{Q}_p^\times)^2 \cong \mathbf{Z}/2 \times \mathbf{Z}/2$. Hint: $\mathbf{Q}_p^\times = p^{\mathbf{Z}} \times \mathbf{Z}_p^\times$; now use parts a) and b).

   d) Deduce that there are only 3 quadratic extensions of $\mathbf{Q}_p$. Hint: a quadratic extension of a field k is of the form $k(\sqrt{a})$ for some $a \in k^\times/(k^\times)^2$.

2. a) Show that $\mathbf{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbf{Z}_2)$.

   b) Show that every $x \in 1 + 8\mathbf{Z}_2$ is a square in $\mathbf{Z}_2^\times$.

   c) Show that $\mathbf{Q}_2^\times/(\mathbf{Q}_2^\times)^2 \cong \mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2$. Find a complete set of representativesin $\mathbf{Q}$ for this quotient group.

   d) Find all 7 quadratic extensions of $\mathbf{Q}_2$.

3. Suppose $p > 2$. Let

   $$F(x_1, ..., x_s) = c_1 x_1^2 + \cdots + c_s x_s^2, \quad c_i \in \mathbf{Q}_p.$$

   Show that if $s \geq 5$ then there is a nonzero solution of $F(x_1, ..., x_s) = 0$ in $\mathbf{Q}_p$. This can be done as follows:

   a) Show that it suffices to assume F is of the form

   $$a_1 x_1^2 + \cdots a_r x_r^2 + p(a_{r+1} x_{r+1}^2 + \cdots + a_s x_s^2), \quad a_i \in \mathbf{Z}_p^\times.$$

   Hint: write $c_i = p^{k_i} a_i$ with $a_i \in \mathbf{Z}_p^\times$ and absorb as much of the p_i^k into the x_i as possible.

   b) Continuing with the notation from a), show that either $a_1 x_1^2 + \cdots + a_r x_r^2 = 0$ or $a_{r+1} x_{r+1}^2 + \cdots + a_s x_s^2 = 0$ has a nonzero solution in $\mathbf{Q}_p$. Hint: either $r \geq 3$ or $s - r \geq 3$; use Chevalley+Warning and Hensel's Lemma.

4. Suppose $d \geq 2$ and $p \nmid d$. Show that if $s \geq d^2 + 1$ then any diagonal equation

   $$c_1 x_1^d + \cdots + c_s x_s^d = 0$$

   has a nonzero solution in $\mathbf{Q}_p$. Hint: generalize the method used for $d = 2$ case.

5. This exercise completes the proof that any quadratic form over $\mathbf{Q}_p$ in five or more variables has a non-zero solution in $\mathbf{Q}_p$. The remaining unproved case is $p = 2$.

   Let

   $$F(x_1, ..., x_s) = c_1 x_1^2 + c_2 x_2^2 + \cdots + c_s x_x^s, \quad c_i \in \mathbf{Q}_2, \quad c_1 \cdots c_s \neq 0.$$

Show that if $s \geq 5$, then $F(x_1, ..., x_s) = 0$ has a nonzero solution in $\mathbf{Q}_2$. This can be done as follows:

a) By writing $c_i = p^{k_i} u_i$ with $u_i \in \mathbf{Z}_2^\times$ and considering the parity of the k_i, after possibly renumbering the variables, reduce to the case where

$$c_i \in \mathbf{Z}_2^\times \text{ if } i \leq r \quad \text{and} \quad c_j \in 2\mathbf{Z}_2^\times \text{ if } j > r$$

for some $r \geq 3$.

b) Show that if $a, b, x, y \in \mathbf{Z}_2^\times$ then $ax^2 + by^2 \in 2\mathbf{Z}_2$.

c) Now argue as follows: Choose $x_1, x_2 \in \mathbf{Z}_2^\times$.

(i) Show that if $c_1x_1^2 + c_2x_2^2 \in 8\mathbf{Z}_2$ then a nonzero solution in $\mathbf{Z}_2$ exists.

(ii) Suppose then that $c_1x_1^2 + c_2x_2^2 \in 4\mathbf{Z}_2^\times$. Show that one can choose $x_3 \in 2\mathbf{Z}_2$ such that $c_1x_1^2 + c_2x_2^2 + c_3x_3^2 \in 8\mathbf{Z}_2$; deduce that a solution in $\mathbf{Z}_2$ exists.

(iii) Suppose then that $c_1x_1^2 + c_2x_2^2 \in 2\mathbf{Z}_2^\times$. Consider two cases: $r \leq 4$ and $r \geq 5$.

(iiia) If $r \leq 4$, choose $x_5 \in \mathbf{Z}_2^\times$. Show that $c_1x_1^2 + c_2x_2^2 + c_5x_5^2 \in 4\mathbf{Z}_2$. If it belongs to $8\mathbf{Z}_2$, then show that there is a nonzero solution in $\mathbf{Z}_2$. If not, then show that there is $x_3 \in 2\mathbf{Z}_2^\times$ such that $c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_5x_5^2 \in 8\mathbf{Z}_2$; deduce that there is then a nonzero solution in $\mathbf{Z}_2$.

(iiib) If $r = 5$, applying the same arguments to $c_3x_3^2 + c_4x_4^2$, you may assume that there is $x_1, x_2, x_3, x_4 \in \mathbf{Z}_2^\times$ such that $c_1x_1^2 + c_2x_2^2, c_3x_3^2 + c_4x_4^2 \in 2\mathbf{Z}_2^\times$. Then $c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_4^2 \in 4\mathbf{Z}_2$. If the sum is actually in $8\mathbf{Z}_2$, then there is a nonzero solution, and if not then show there is $x_5 \in 2\mathbf{Z}_2$ such that $c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_4^2 + c_5x_x^5 \in 8\mathbf{Z}_2$ and conclude that a nonzero solution in $\mathbf{Z}_2$ exists.

6. Let $x \in \mathbf{Z}_p$. Show that for any $n > 0$ there exists a non-negative integer y such that $x - y \in p^n\mathbf{Z}_p$.

7. For each integer m, define a function $\binom{x}{m} : \mathbf{Z}_p \to \mathbf{Q}_p$ by

$$\binom{z}{m} = \frac{x(x-1)(x-2)\cdots(x-m+1)}{m!}.$$

Prove that for all $x \in \mathbf{Z}_p$, $\binom{x}{m} \in \mathbf{Z}_p$. Hint: use that this is true for $x \geq 0$ an integer and use that any element of $\mathbf{Z}_p$ can be closely approximated by a non-negative integer, as in the preceding problem.

8. Show that an infinite series $\sum_{n=0}^{\infty} a_n$, $a_n \in \mathbf{Q}_p$, converges p-adically if and only if $|a_n|_p \to 0$.

9. a) Show that the series $\exp_p(x) := \sum_{n=0}^{\infty} x^n/n!$ converges p-adically for all $x \in p\mathbf{Z}_p$ if p is odd and for all $x \in 4\mathbf{Z}_2$ if $p = 2$. Hint: the power of p dividing $n!$ is $[n/p] + [n/p^2] + [n/p^2] + \cdots < n/(p-1)$.

b) Explain why $\exp_p(x) \in 1 + p\mathbf{Z}_p$.

c) Explain why $\exp_p(x + y) = \exp_p(x)\exp_p(y)$.

10. a) Show that the series $\log_p(1+x) := \sum_{n=1}^{\infty}(-1)^{n-1}x^n/n$ converges p-adically for all $x \in p\mathbf{Z}_p$.

b) Verify that $\log_p \circ \exp_p(x) = x$.