Kontsevich's Quantization

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3º ano LMAC
Instituto Superior Técnico

Encontro Nacional NTM, 2008

For his contributions to algebraic geometry, topology, and mathematical physics, including the proof of Witten's conjecture of intersection numbers in moduli spaces of stable curves, construction of the universal Vassiliev invariant of knots, and **formal quantization of Poisson manifolds**.

Yuri I. Manin, chairman of the Fieds Medal Comittee,
at the Opening Ceremony of the ICM[1],
18/08/1998

[1] International Congress of Mathematicians

Structure

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where

$$\{f,g\} = \sum_{i=1}^{N}\left[\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right]$$

This bracket is a bidifferential operator with the following properties:

- Anti-symmetry: $\{f, g\} = -\{g, f\}$,
- Leibniz's identity: $\{f, gh\} = \{f, g\}h + g\{f, h\}$,
- Jacobi's identity: $\{f, \{g, h\}\} = \{h, \{f, g\}\} + \{g, \{h, f\}\}$.

Definition

A Poisson manifold $(M, \{.,.\})$ is a differentiable manifold M such that $C^\infty(M)$ has a bilinear map $\{.,.\}$ with the following properties:

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$\{.,.\}$ is called a Poisson bracket.

In local coordinates x_i around a point $S(\mathbf{q})$:

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where

$$\Pi = (\alpha_{ij}) = \begin{pmatrix} 0 & I_{k\times k} & 0 \\ -I_{k\times k} & 0 & 0 \\ 0 & 0 & S_{s\times s} \end{pmatrix},$$

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Symplectic manifolds are special cases of Poisson Manifolds in which $S = 0$. The symplectic form is:

$$\omega(x) = \sum_{i,j=1}^{N} \alpha^{ij}(x) dx_i \wedge dx_j,$$

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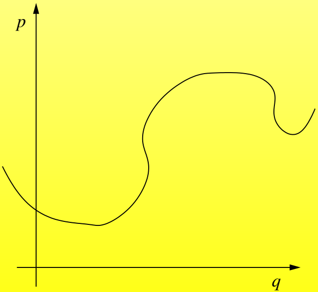
- The state of the system is its coordinates in the phase space, ex: position and momenta coordinates,
- An observable is a function $f : M \to \mathbb{R}$, ex: $E(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\frac{|\mathbf{p}|^2}{m} + V(\mathbf{q})$.



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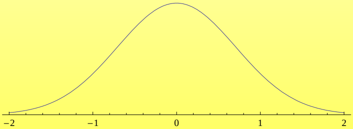

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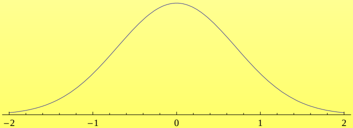

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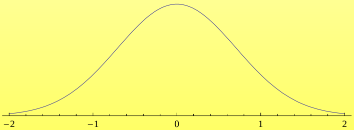

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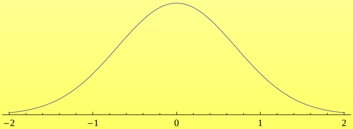

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Radical changes:

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- In general observables don't commute, $AB \neq BA$!!!

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In the so-called Schrödinger representation one has:

- $\hat{q}_i f = q_i f$
- $\hat{p}_i f = -i\hbar \frac{\partial f}{\partial q_i}$

Note that $\{q_i, p_i\} = 1$ and $[\hat{q}_i, \hat{p}_i] = i\hbar$.

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Definition

A quantization is a linear map assigning to each $f \in C^\infty(M)$ a linear operator $\hat{f}$ in a Hilbert space, $\mathcal{H}$, such that:

- $\hat{1} = Id$;
- $[\hat{f}, \hat{g}] = \hbar i \widehat{\{f, g\}} + O(\hbar^2)$.

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3. $(f \star_\hbar g) \star_\hbar h = f \star_\hbar (g \star_\hbar h)$;
4. $\left(\sum_{k=0}^{\infty} f_k \hbar^k\right) \star_\hbar \left(\sum_{l=0}^{\infty} g_l \hbar^l\right) = \sum_{k,l=0}^{\infty} (f_k \star_\hbar g_l)\hbar^{k+l}$;

Example

Weyls quantization extends the quantization (in $\mathbb{R}^{2N}$)

- $q_i \longrightarrow \hat{q}_i$
- $p_i \longrightarrow \hat{p}_i$

by symmetry:

$$f \mapsto \hat{f} \equiv \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} e^{-i(q.\hat{p}+p.\hat{q})} (\mathcal{F}f)(q,p) dqdp$$

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It gives rise to Moyal's $\star_\hbar$-product:

$$f \star_\hbar g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2} \sum_{j=1}^{N} \left(\frac{\partial}{\partial q_1^j} \frac{\partial}{\partial p_2^j} - \frac{\partial}{\partial p_1^j} \frac{\partial}{\partial q_2^j} \right) \right)^n f(q_1,p_1) g(q_2,p_2) \Bigg|_{\substack{p_1=p_2=p \\ q_1=q_2=q}}$$

When does there exist Deformation Quantization?

- On every symplectic manifold (De Wilde e Lecomte, 1983)
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Was it divine inspiration?

In 1999 Cattaneo and Felder found a (heuristic) way to obtain this formula, through the use of Path Integral.
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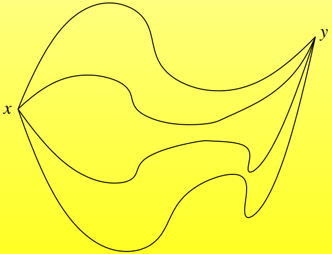
An integral on the space of paths:

$$\int_{\mathcal{P}_{xy}^{t}} f(\gamma)d\gamma$$



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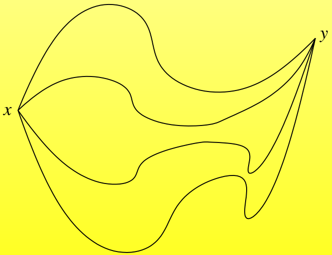
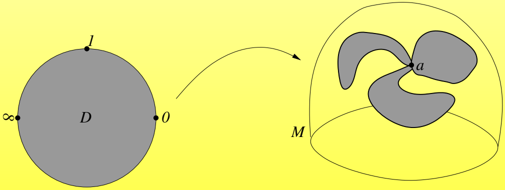


In most cases it is ill-defined.

The derivation in 4 (big!) steps:

- Start by rewriting Moyal's product in $\mathbb{R}^{2N}$ as a path integral:

$$(f \star_\hbar g)(a) = \int_{\left\{ \substack{\phi : D \to \mathbb{R}^{2N}, \\ \phi(\infty) = a} \right\}} f(\phi(0)) g(\phi(1)) e^{\frac{4i}{\hbar} \int_D \phi^* \omega} D\phi$$



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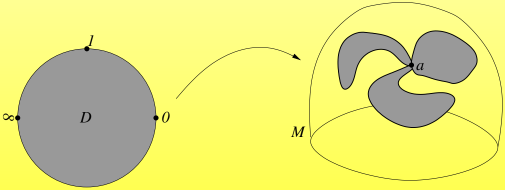

- Use the same formula but on a symplectic manifold, (M, ω):

$$(f \star_\hbar g)(a) = \int_{\left\{\substack{\phi: D \to M, \\ \phi(\infty)=a}\right\}} f(\phi(0))g(\phi(1))e^{\frac{4i}{\hbar}\int_D \phi^*\omega} D\phi$$

- Apply the analog of a Fourier transform on each point to get

$$(f \star_\hbar g)(a) = \int_{\left\{\substack{\phi : D \to M, \\ \phi(\infty) = a}\right\}} \int_{\eta \in \phi^*(T^*M) \otimes T^*D} f(\phi(0)) g(\phi(1)) e^{\frac{16i}{\hbar} S[\phi,\eta]} D\eta D\phi,$$

where

$$S[\phi, \eta] = \int_D \sum_{i=1}^{2N} \eta_i \wedge d\phi_i + \sum_{i,j=1}^{2N} \alpha_{ij} \circ \phi \, \eta_i \wedge \eta_j$$

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- This formula, which aparently can be used in any Poisson Manifold (despite being ill-defined), gives Kontsevich's formula after the use of QFT methods and semi-classical approximations.

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