LECTURES ON LOOP GROUPS

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Date: June 14, 2018.

1. **Lecture I, Background and Overview**

Complex and Unitary forms:

We set up notation by letting K denote a compact, simply connected, simple Lie group. For example $K = SU(n)$. By G we shall mean the complexification of K. So, up to isomorphism, G is the unique simple complex algebraic group which includes K as a maximal compact subgroup. So in our example $G = SL_n(\mathbb{C})$.

In fact, there is an anti-linear involution σ on G called the Cartan involution whose fixed points is K. In our example of $SL_n(\mathbb{C})$, this involution is given by $\sigma(A) = (A^*)^{-1}$.

Algebraic and smooth Loop groups:

We define two types of loop groups $L_{sm}K$ and $L_{alg}K$. The group $L_{sm}K$ is the (pointwise) group of smooth maps from S^1 to K in the compact open topology. The group $L_{alg}K$ may be defined as the subgroup of maps with finite Fourier expansions under all unitary representations of K. However, the topology on $L_{alg}K$ is given by the direct limit topology induced by the compact sub-spaces of $L_{sm}K$ that consist of maps of increasing degree under some faithful unitary representation of K.

Alternatively, to define $L_{alg}K$, we may mimic the above story by starting with the complex form $L_{alg}G = G(\mathbb{C}[z, z^{-1}])$ of $\mathbb{C}^*$-valued points in G, and then take fixed points under the (loop) Cartan involution $\hat{\sigma}(A)(z) = \sigma\{A(\sigma_1(z))\}$, where σ_1 is the automorphism of $\mathbb{C}^*$ sending z to $\overline{z}^{-1}$.

Claim 1.1. *Let $\hat{\sigma}$ denote the Cartan involution on $L_{alg}G$, then the set of fixed points of $\hat{\sigma}$ is exactly the set of maps $A(z) \in L_{alg}G$ such that $A(S^1) \subseteq K$. In particular, seen as an element in $L_{sm}K$, the map $A(z)$ has a finite fourier expansion under any unitary representation of K. Conversely, any map $A(z) \in L_{sm}K$ that has a finite fourier expansion under all unitary representations of K, extends uniquely to an element in $L_{alg}G$.*

Proof. See Appendix. □

Theorem 1.2. *The map $L_{alg}K \to L_{sm}K$ is a homotopy equivalence. Hence the homotopy/homology of the Loop group is independent of the type of loops.*

Proof. See Appendix. □

Remark 1.3. *Henceforth, we will denote smooth loops simply by LK and algebraic groups by $L_{alg}K$. The properties these groups share are very similar and in the sequel, we will point out how properties of one can be derived from those of the other.*

The central extension, the extended Loop group and the Affine Weyl group:

Notice that the map $LK \to K$ given by evaluating at $1 \in S^1$ is split by the constant loops, hence $LK = K \ltimes \Omega K$. It follows that $H^2(LK, \mathbb{Z}) = \mathbb{Z}$. Let $\tilde{L}K$ denote the principal S^1 bundle over LK representing the generator of $H^2(LK, \mathbb{Z})$.

Claim 1.4. *The space $\tilde{L}K$ has the structure of a universal S^1-central extension of LK. In particular, $\tilde{L}K$ is 2-connected and any nontrivial S^1-central extension of LK is a finite quotient of $\tilde{L}K$. The inclusion $L_{alg}K \subset LK$ induces a similar universal central extension $\tilde{L}_{alg}K$ called the Affine Kac-Moody group.*

Proof. See Appendix. □

Let $\mathbb{T}$ denote the circle acting on LK by rotation: $e^{i\theta}. A(z) = A(e^{i\theta}z)$. Then the $\mathbb{T}$ action on LK lifts to $\tilde{L}K$, giving rise to the group $\mathbb{T} \ltimes \tilde{L}K$. As before $\mathbb{T} \ltimes \tilde{L}_{alg}K$ is called the extended Affine Kac-Moody group.

Let T denote the maximal torus of K, and let $W = N_K(T)/T$ be the finite Weyl group. Then $\mathbb{T} \times S^1 \times T \subset \mathbb{T} \ltimes \tilde{L}K$ is a maximal torus, and its Weyl group $\tilde{W}$ is called the Affine Weyl group. This group is isomorphic to $W \ltimes \pi_1(T)$.

The extended Affine Lie algebra:

Let $\mathfrak{g}$ denote the (complex) Lie algebra of G (the complexification of K). Then the Lie algebra of $L_{sm}G$ can easily be seen to be the space of smooth maps form $\mathbb{C}^*$ to $\mathfrak{g}$. In the case of algebraic loops, one formally defines the Lie algebra of $L_{alg}G$ to be $L_{alg}\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$. The extended Affine Lie algebra $\hat{\mathfrak{g}}$ is given by:

$$\hat{\mathfrak{g}} = \mathbb{C}\langle d\rangle \oplus \mathbb{C}\langle k\rangle \oplus \mathfrak{g}[z, z^{-1}], \quad [\alpha z^n, \beta z^m] = [\alpha, \beta] z^{m+n} + k\, \langle \alpha, \beta\rangle Res(z^n d(z^m)),\ [d, \alpha z^n] = n\alpha z^n$$

with $\langle\ ,\ \rangle$ denoting the Cartan-Killing form on $\mathfrak{g}$, $\mathbb{C}\langle d\rangle$ denoting the complexified Lie algebra of $\mathbb{T}$, and $\mathbb{C}\langle k\rangle$ denoting the complexified Lie algebra of the center S^1.

Remark 1.5. *The cocycle representing the center that we have chosen is the opposite of the standard convention as in* [K]*. On the other hand, it is consistent with* [PS]*. We make this choice since with the standard convention, the positive energy representations, which correspond to lowest weight representations, have negative level (see below).*

Unitary representations, level and energy:

Given a unitary representation of $\mathbb{T} \ltimes \tilde{L}K$ in a Hilbert space $\mathcal{H}$, we may restrict to $\mathbb{T}$ to obtain a dense inclusion:

$$\mathcal{H}_{alg} := \bigoplus_n q^n \mathcal{H}_n \subseteq \mathcal{H}$$

where q is the fundamental character of $\mathbb{T}$, and $\mathcal{H}_n$ is the corresponding isotypical summand. We say $\mathcal{H}$ is a representation of positive (resp. negative) energy if each $\mathcal{H}_n$ is finite dimensional and $\mathcal{H}_n = 0$ for $n < 0$ (resp. $n > 0$). Note that if $\mathcal{H}_n = 0$ for sufficiently small (resp. large) values of n, we may tensor $\mathcal{H}$ by a suitable character of $\mathbb{T}$ to make it positive (resp. negative) energy.

If $\mathcal{H}$ is an irreducible unitary representation of $\mathbb{T} \ltimes \tilde{L}K$, then by Schur's lemma, the central circle acts by a fixed character μ on $\mathcal{H}$. The character μ is called the level of $\mathcal{H}$. There are finitely many irreducible positive (resp. negative) energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of a fixed level. Indeed, the category of unitary positive energy (resp. negative energy) representations of $\mathbb{T} \ltimes \tilde{L}K$ of a given level, is semisimple.

Similarly, one may define the category of unitary positive (resp. negative energy) representations of $\mathbb{T} \ltimes \tilde{L}_{alg}K$. Restriction of any such representation of $\mathbb{T} \ltimes \tilde{L}K$ to $\mathbb{T} \ltimes \tilde{L}_{alg}K$ establishes an equivalence of these two categories. Given a positive (resp. negative energy) representation $\mathcal{H}$ of $\mathbb{T} \ltimes \tilde{L}K$, the dense subspace $\mathcal{H}_{alg}$ is invariant under $\mathbb{T} \ltimes \tilde{L}_{alg}K$ and serves as a (pre)-Hilbert space completing to $\mathcal{H}$.

Tensor products and Fusion (positive energy formulation):

Let $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ denote two irreducible positive energy representations of respective levels λ and μ. Then the (vector space) tensor product $\mathcal{H}_\lambda \otimes \mathcal{H}_\mu$ can be decomposed into an infinite sum of positive energy representations of level $\lambda+\mu$. Hence, the naive tensor product does not preserve the level. Now let A_λ denote the Grothendieck group of the category of positive energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of a fixed level λ and finite composition series. Then A_λ is a finitely generated free abelian group that supports a (subtle) commutative product called the fusion product. The induced ring structure on A_λ is called the fusion ring or Verlinde ring. The unit in this ring is called the Vacuum representation of level λ.

Remark 1.6. *We will show in the third lecture that the Hilbert completion of the exterior algebra $\Lambda^*(\mathbb{C}^n[z]) \otimes \Lambda^*(z\overline{\mathbb{C}}^n[z])$ is an irreducible positive energy representation of $\mathbb{T} \ltimes \tilde{L}U(n)$ of level one. Indeed, this representation contains all level one representations of $\mathbb{T} \ltimes \tilde{L}SU(n)$. This is also known as the Fermionic Fock space representation.*

2. **Lecture II, Structure theory of the Affine Lie algebra**

Review of the Finite dimensional Theory:

Recall that the finite dimensional Lie algebra $\mathfrak{g}$ admits a symmetric, bilinear, non-degenerate, $\mathfrak{g}$-invariant (Cartan-Killing) form given by: $\langle \alpha, \beta \rangle = \mathrm{Tr}\, ad(\alpha)\, ad(\beta)$.

Let $\mathfrak{h}$ be the (Cartan) maximal abelian sub-algebra of $\mathfrak{g}$ given by the complexification of the Lie algebra of T. The Cartan-Killing form restricts to a non-degenerate form on $\mathfrak{h}$.

Under the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$, one has the eigenspace decomposition called triangular decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \eta_+ \oplus \eta_-, \quad \eta_\pm = \oplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}$$

where $\Delta_+ \subset \mathfrak{h}^*$ is the set of (non-zero) positive roots defined by the property $[h, x] = \pm\alpha(h)\, x$ for $h \in \mathfrak{h}, x \in \mathfrak{g}_{\pm\alpha}$. The spaces $\mathfrak{g}_{\pm\alpha}$ are called the root spaces of $\mathfrak{g}$.

Each root space $\mathfrak{g}_{\pm\alpha}$ is one dimensional generated by elements $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$ called root vectors. The elements $h_\alpha = [e_\alpha, f_\alpha]$ are called co-roots and span $\mathfrak{h}$. In addition, $h_\alpha = 2\alpha^*/\langle \alpha, \alpha \rangle$.

Let $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ be a real form (so that each α takes real values on $\mathfrak{h}_\mathbb{R}$). Then $h_\alpha \in \mathfrak{h}_\mathbb{R}$. We define the anti-dominant Weyl chamber of $\mathfrak{h}^*$ to be the polyhedral cone:

$$C = \{\beta \in \mathfrak{h}_\mathbb{R}^* \,|\, \beta(h_\alpha) \leq 0, \,\forall\, \alpha \in \Delta_+\}$$

There exist a set of positive roots $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta_+$ (called the simple roots) such they are linearly independent in $\mathfrak{h}_\mathbb{R}^*$ and such that any $\alpha \in \Delta_+$ is a non-negative integral linear combination of the simple roots. There is a unique positive root $\alpha_0 \in \Delta_+$ (called the highest root) which is the largest in the sense that $\alpha_0 - \alpha$ is a non-negative integral combination of the simple roots for any $\alpha \in \Delta_+$. By convention, we normalize the Cartan-Killing form so that $\langle \alpha_0, \alpha_0 \rangle = 2$.

The Weyl group is generated by the set $\{r_{\alpha_1}, r_{\alpha_2}, \ldots, r_{\alpha_n}\}$ of reflections about the walls of the Weyl chamber given by the planes $\beta(h_{\alpha_i}) = 0$. The explicit formula for the action of r_{α_i} on $\beta \in \mathfrak{h}_\mathbb{R}^*$ is given by $r_{\alpha_i}(\beta) = \beta - \beta(h_{\alpha_i})\, \alpha_i$.

Example 2.1. Consider the example of $\mathfrak{g} = sl_3(\mathbb{C})$ of traceless 3×3 complex matrices. Then $\mathfrak{h}$ is given by the diagonal matrices and $\eta_\pm$ is the space of upper-triangular (resp. lower-triangular) matrices. The positive roots are given by $\alpha_1, \alpha_2, \alpha_0$ with $\alpha_0 = \alpha_1 + \alpha_2$ defined as follows: For a diagonal matrix h with entries (x_1, x_2, x_3) we have:

$$\alpha_1(h) = x_1 - x_2, \quad \alpha_2(h) = x_2 - x_3, \quad \alpha_0(h) = x_1 - x_3.$$

The classes h_α are given by $h_{\alpha_1} = (1, -1, 0)$, $h_{\alpha_2} = (0, 1, -1)$ and $h_{\alpha_0} = (1, 0, -1)$ in the diagonal representation. In addition, we have:

$$e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{\alpha_0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with } f_\alpha = e_\alpha^T$$

The Weyl group is isomorphic to the symmetric group Σ_3 generated by two reflections about the walls of C (that subtend an angle of $\pi/3$).

The Affine Lie algebra:

Recall that the Affine Lie algebra is given by:

$$\hat{\mathfrak{g}} = \mathbb{C}\langle d \rangle \oplus \mathbb{C}\langle k \rangle \oplus \mathfrak{g}[z, z^{-1}], \quad [f, g](z) = [f(z), g(z)] + k\, Res\langle f, dg \rangle, \quad [d, f] = z\frac{\partial f}{\partial z}.$$

The cartan involution on $\hat{\mathfrak{g}}$ is given by:

$$\hat{\sigma}(\alpha\, z^n) = \sigma(\alpha)\, z^{-n}, \quad \hat{\sigma}(k) = -k, \quad \hat{\sigma}(d) = -d.$$

Let $\hat{\mathfrak{k}}$ denote the fixed points of $\hat{\sigma}$. We formally think of $\hat{\mathfrak{k}}$ as the Lie algebra of $\mathbb{T} \ltimes \tilde{L}_{alg}K$.

Identifying $\mathfrak{h}$ with $\mathfrak{h} \otimes 1 \subset \hat{\mathfrak{g}}$, we have the Cartan sub-algebra and its real form:

$$\hat{\mathfrak{h}} = \mathbb{C}\langle d \rangle \oplus \mathbb{C}\langle k \rangle \oplus \mathfrak{h}, \quad \hat{\mathfrak{h}}_\mathbb{R} = \mathbb{R}\langle d \rangle \oplus \mathbb{R}\langle k \rangle \oplus \mathfrak{h}_\mathbb{R}, \quad \hat{\mathfrak{h}}^*_\mathbb{R} = \mathbb{R}\langle \delta \rangle \oplus \mathbb{R}\langle \Lambda \rangle \oplus \mathfrak{h}^*_\mathbb{R}$$

where $\delta, \Lambda \in \hat{\mathfrak{h}}^*_\mathbb{R}$ denote the elements dual to d and k under the above decomposition. The triangular decomposition is given by $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \hat{\eta}_+ \oplus \hat{\eta}_-$ with

$$\hat{\eta}_+ = z\,\mathfrak{g}[z] \,\oplus\, \eta_+ = \{f(z) \in \mathfrak{g}[z] \,|\, f(0) \in \eta_+\}$$
$$\hat{\eta}_- = z^{-1}\mathfrak{g}[z^{-1}] \,\oplus\, \eta_- = \{f(z) \in \mathfrak{g}[z^{-1}] \,|\, f(\infty) \in \eta_-\}$$

We may describe the positive roots of the $\hat{\mathfrak{h}}$-action on $\hat{\eta}_+$:

$\hat{\Delta}_+ = \{(n\delta + \epsilon\alpha) \in \hat{\mathfrak{h}}^*_\mathbb{R} \,|\, n \in \mathbb{N}, \alpha \in \Delta_+ \text{ with } n > 0 \,\text{and}\, \epsilon \in \{-1, 0, 1\} \text{ OR } n = 0, \text{ and } \epsilon = 1\}$

The individual root spaces are given by:

$$\hat{\mathfrak{g}}_\alpha = \mathbb{C}\langle e_\alpha \rangle, \quad \hat{\mathfrak{g}}_{n\delta+\alpha} = \mathbb{C}\langle e_\alpha\, z^n \rangle, \quad \hat{\mathfrak{g}}_{n\delta-\alpha} = \mathbb{C}\langle f_\alpha\, z^n \rangle, \quad \hat{\mathfrak{g}}_{n\delta} = \mathfrak{h}\, z^n$$

The roots $\{n\delta\}$ are called the imaginary roots and are the only positive roots with multiplicity bigger than one. Notice that we have a collection of simple roots given by:

$$\{\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_{n+1}\} = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \delta - \alpha_0\}$$

The corresponding root-vectors are given by:

$$e_{\hat{\alpha}_i} = e_{\alpha_i}, \quad f_{\hat{\alpha}_i} = f_{\alpha_i} \quad \text{for} \quad i \leq n, \quad \text{and} \quad e_{\hat{\alpha}_{n+1}} = f_{\alpha_0}\, z, \quad f_{\hat{\alpha}_{n+1}} = e_{\alpha_0}\, z^{-1}$$

The simple co-roots $h_{\hat{\alpha}_i}$ are given by the formula $h_{\hat{\alpha}_i} = [e_{\hat{\alpha}_i}, f_{\hat{\alpha}_i}]$, and so:

$$h_{\hat{\alpha}_i} = h_{\alpha_i} \quad \text{for} \quad i \leq n, \quad \text{and} \quad h_{\hat{\alpha}_{n+1}} = -(k + h_{\alpha_0})$$

The subgroups of the Affine Kac-Moody group corresponding to the triangular decomposition:

For the algebraic group G, the subgroups corresponding to the subalgebras $\mathfrak{h}, \eta_+, \eta_-$ are H, U_+, U_-, where H is called the Cartan subgroup (it is the complexification of the maximal torus of K), and $U_\pm$ are called the positive (resp. negative) unipotent subgroups. In the example of $SL_n(\mathbb{C})$, the group H is the diagonal matrices, and $U_\pm$ are the upper (resp. lower) triangular matrices.

There are corresponding subgroups of the complex form $\mathbb{C}^* \ltimes \tilde{G}(\mathbb{C}[z, z^{-1}])$ that correspond to $\hat{\mathfrak{h}}, \hat{\eta}_+, \hat{\eta}_-$ given by $\hat{H}, \hat{U}_+, \hat{U}_-$ resp., with $\hat{H} = \mathbb{C}^*(d) \times \mathbb{C}^*(k) \times H$ and:

$$\hat{U}_+ = \{f(z) \in G(\mathbb{C}[z]) \,|\, f(0) \in U_+\}, \quad \hat{U}_- = \{f(z) \in G(\mathbb{C}[z^{-1}]) \,|\, f(\infty) \in U_-\}$$

We may define the Borel sub-algebras by $\hat{b}_\pm = \hat{\mathfrak{h}} \oplus \hat{\eta}_\pm$. The corresponding groups generated by $\hat{H}$ and $\hat{U}_\pm$ are called the Borel subgroups $\hat{B}_\pm$.

The Affine Weyl group:

On $\hat{\mathfrak{h}}_\mathbb{R}$ there is a non-degenerate, symmetric form (which is not positive definite) given by extending the Cartan-Killing form on $\mathfrak{h}_\mathbb{R}$, demanding that $\mathbb{R}\langle d\rangle \oplus \mathbb{R}\langle k\rangle$ be orthogonal to $\mathfrak{h}_\mathbb{R}$ and the formulas: $\langle d, k\rangle = -1$ (the negative sign corresponds to the opposite orientation on the center compared to the standard orientation), and $\langle d, d\rangle = \langle k, k\rangle = 0$. This form induces a dual form on $\hat{\mathfrak{h}}^*_\mathbb{R}$ and the Affine Weyl group $\hat{W}$ is generated by reflections $r_{\hat{\alpha}_i}$ about the walls $\beta(h_{\hat{\alpha}_i}) = 0$ of the anti-dominant Affine Weyl chamber $\hat{C}$ in $\hat{\mathfrak{h}}^*_\mathbb{R}$. We have the following formulas:

$$r_{\hat{\alpha}_i}(\alpha) = r_{\alpha_i}(\alpha) \quad i \leq n, \qquad r_{\hat{\alpha}_{n+1}}(\alpha) = r_{\alpha_0}(\alpha) + \alpha(h_{\alpha_0})\,\delta$$

$$r_{\hat{\alpha}_i}(\Lambda) = \Lambda \quad i \leq n, \qquad r_{\hat{\alpha}_{n+1}}(\Lambda) = \Lambda + \delta - \alpha_0$$

$$r_{\hat{\alpha}_i}(\delta) = \delta \qquad i \leq n+1$$

We observe two facts. Firstly, one observes that $\hat{W}$ preserves the affine slices (called levels) given by vectors of the form $l\Lambda \oplus \mathbb{R}\langle\delta\rangle \oplus \mathfrak{h}^*_\mathbb{R}$. Secondly, the action of $\hat{W}$ is well-defined and faithful on the vector space $\hat{\mathfrak{h}}^*_\mathbb{R}/\mathbb{R}\langle\delta\rangle$. Working modulo δ, we have the following formulas:

$$r_{\hat{\alpha}_{n+1}} r_{\alpha_0}(\Lambda) = \Lambda - \alpha_0, \quad r_{\hat{\alpha}_{n+1}} r_{\alpha_0}(\alpha) = \alpha$$

It follows that the normal subgroup of $\hat{W}$ generated by $r_{\hat{\alpha}_{n+1}} r_{\alpha_0}$ is isomorphic to the lattice I in $\mathfrak{h}^*_\mathbb{R}$ generated by the W orbit of α_0. From this observation we see that

$$\hat{W} = \langle r_{\hat{\alpha}_1}, r_{\hat{\alpha}_2}, \ldots, r_{\hat{\alpha}_n}, r_{\hat{\alpha}_{n+1}}\rangle = \langle r_{\hat{\alpha}_1}, r_{\hat{\alpha}_2}, \ldots, r_{\hat{\alpha}_n}, r_{\hat{\alpha}_{n+1}} r_{\alpha_0}\rangle \cong W \ltimes \mathrm{I}$$

The Affine Alcove:

A very important action of the Affine Weyl group is given by its action on the affine subspace described above: $l\Lambda \oplus \mathfrak{h}_{\mathbb{R}}^*$ where we are working modulo δ. Under this action, the subgroup I acts by translation by l I, and W acts through its action on $\mathfrak{h}_{\mathbb{R}}^*$. It is easy to see that the fundamental domain of this action is contained in the anti-dominant Affine Weyl chamber $\hat{C}$, and by projecting to $\mathfrak{h}_{\mathbb{R}}^*$ it can be identified with the (scaled) anti-dominant Affine Alcove lA, where

$$l\mathrm{A} = \{\beta \in C \,|\, \beta(h_{\alpha_0}) \geq -l\}$$

It is important to take note that lA is non-empty if and only if $l \geq 0$.

3. Lecture III, The theory of positive energy representations

Let $\mathcal{H}$ be a level l, positive energy representation of $\mathbb{T} \ltimes \tilde{L}K$. We decompose it under the compact subgroup $\mathbb{T} \times S^1 \times K$ to get a dense subspace of $\mathcal{H}$:

$$\mathcal{H}_{alg} = \bigoplus_n q^n V_n$$

where V_n are finite dimensional representations of $S^1 \times K$ of level l. By assumption $V_n = 0$ for $n < 0$. The subspace V_0 is called the states of zero energy, and we assume that $V_0 \neq 0$. By the assumption of positive energy, all the negative root vectors of $\hat{\mathfrak{g}}$ not in $\mathfrak{g}$ act trivially on V_0. And hence we have a non-zero map:

$$\pi : \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{b}_- + \mathfrak{g})} V_0 \longrightarrow \mathcal{H}_{alg}$$

where the action of $(\hat{b}_- + \mathfrak{g})$ on V_0 factors through the reductive sub-algebra $(\hat{\mathfrak{h}} + \mathfrak{g})$. By the triangular decomposition and the PBW theorem, we observe that $\mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{b}_- + \mathfrak{g})} V_0$ is isomorphic as an $\hat{\mathfrak{h}}$-module to $\mathcal{U}(\hat{\mu}_+) \otimes V_0$, where $\hat{\mu}_+ \subset \hat{\eta}_+$ is the sub-algebra generated by all positive root vectors not in $\mathfrak{g}$. Now let us decompose V_0 into a sum of irreducibles with multiplicity: $V_0 = \oplus V_{\tau_i}$, where V_{τ_i} is an irreducible with lowest weight $\tau_i \in \hat{\mathfrak{h}}_{\mathbb{R}}^*$ of the form $l\Lambda + \lambda_i$ with λ_i a weight in the anti-dominant Weyl chamber C of $\mathfrak{g}$. In addition, since $\mathcal{H}$ has an action of the Affine Weyl group, we observe that $r_{\hat{\alpha}_{n+1}}$ acts on the weight $l\Lambda + \lambda_i$ to yield another weight in $\mathcal{H}_{alg}$. It follows from the explicit formulas that $\lambda_i(h_{\alpha_0}) + l \geq 0$ and hence that λ_i belongs to the anti-dominant Affine Alcove lA.

Now assume that $\mathcal{H}$ is irreducible. It follows that the representation $\mathcal{H}_{alg}$ of the Affine Lie algebra $\hat{\mathfrak{g}}$ is also irreducible and that π is surjective. In addition, we deduce that V_0 must be an irreducible representation of $S^1 \times K$ (since the $\hat{\mathfrak{g}}$ span on V_0 cannot have other vectors of zero-energy). Hence $V_0 = V_\tau$ for some weight $\tau \in \hat{\mathfrak{h}}_{\mathbb{R}}^*$ of the form $l\Lambda + \lambda$ with λ belonging to the anti-dominant Affine Alcove lA. It is easy to see that the positive energy $\hat{\mathfrak{g}}$-module of the form $\mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{b}_- + \mathfrak{g})} V_\tau$ has a unique maximal ideal [K], and so the corresponding irreducible quotient L_τ is an irreducible representation isomorphic to $\mathcal{H}_{alg}$.

L_τ as a dense subspace of holomorphic sections [PS] (Ch 11):

Let $\hat{G}$ denote $\mathbb{C}^* \ltimes \tilde{G}(\mathbb{C}[z, z^{-1}])$. Given $\tau \in l$A, consider the holomorphic line bundle over the (complex) homogeneous space $\hat{G}/\hat{B}_+$ given by $\mathcal{L}_\tau = \hat{G} \times_{\hat{B}_+} \mathbb{C}_\tau$, with $\hat{B}_+$ acting on $\mathbb{C}$ via the character τ as described earlier. We claim that the space of holomorphic sections of $\mathcal{L}_\tau$ contains a dense subspace isomorphic to L_τ. To see this, it is sufficient to show that the space of $\hat{\eta}_-$ invariant vectors is one dimensional, since these invariant vectors index exactly the irreducible lowest energy states inside any (sub) module. We establish this by restricting the bundle to the contractible dense open set equivalent to $\hat{U}_-$ (whose lie algebra is $\hat{\eta}_-$), we see that the only such sections are the constants. In addition, if τ belongs to lA, this constant section can be shown to extend to a global section. Putting together everything mentioned so far yields:

Theorem 3.1. *Taking lowest energy states establishes a bijection between irreducible positive energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of level l, and irreducible representations of K with lowest weight in lA. Here the weight $l\Lambda$ will correspond to the Vacuum under fusion.*

Complete reducibility:

Now returning to an arbitrary positive energy representation $\mathcal{H}$, with zero energy states given by V_0, let $v \in V_0$ be a lowest weight vector with anti-dominant weight τ. Let $p : \mathcal{H}_{alg} \to \mathbb{C}\langle v \rangle$ denote a $\mathbb{T} \times S^1 \times T$-equivariant projection. This extends to a $\mathbb{T} \ltimes \tilde{L}K$-equivariant map:

$$\mathcal{H}_{alg} \longrightarrow \Omega_{hol}(\mathcal{L}_\tau), \qquad \xi \mapsto \;\; f(z) \mapsto p(f(z)^{-1}\xi)$$

where we have identified $\Omega_{hol}(\mathcal{L}_\tau)$ with a suitable completion of the space of (right) $\hat{B}_+$-equivariant polynomial maps from $\hat{G}$ to $\mathbb{C}\langle v \rangle$. Splitting the map given above allows us to prove complete reducibility.

The character formula:

The interpretation of the irreducible L_τ as a dense subspace of holomorphic sections allows us to calculate its character (modulo technical details) via the fixed point formula:

$$\mathrm{Ch}(L_\tau) = \frac{\sum_{w \in \hat{W}} (-1)^w \, e^{w(\tau-\rho)}}{e^{-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{\alpha})^{\dim \hat{\mathfrak{g}}_\alpha}}$$

where $\rho \in \hat{\mathfrak{h}}^*_{\mathbb{R}}$ is the unique weight with the property $\rho(h_{\hat{\alpha}_i}) = 1$ for all $i \leq n+1$. Here, one should think of the expression $(-1)^w e^{w(\rho)-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{\alpha})^{\dim \hat{\mathfrak{g}}_\alpha}$ as the character of the complex Spinor representation for the complex Clifford algebra on the tangent space at the $\mathbb{T} \times S^1 \times T$ fixed point given by $w\hat{B}_+ \in \hat{G}/\hat{B}_+$.

Notice in particular that for the trivial level 0 representation, one has the Weyl-Kac denominator formula:

$$\sum_{w \in \hat{W}} (-1)^w \, e^{\rho - w(\rho)} = \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{\alpha})^{\dim \hat{\mathfrak{g}}_\alpha}$$

An example: The denominator formula for $\mathbb{T} \ltimes \tilde{L}_{alg}SU(2)$:

Consider the case of the group $K = SU(2)$. Let $\hat{\mathfrak{g}}$ denote the Affine Lie algebra with two simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$. The positive roots are given by:

$$\hat{\Delta}_+ = \{n\delta + \hat{\alpha}_1, \;\; n\delta + \hat{\alpha}_2; \quad n \geq 0\} \quad \coprod \quad \{n\delta; \quad n > 0\}, \quad \text{where} \quad \delta = \hat{\alpha}_1 + \hat{\alpha}_2.$$

Here the maximal torus of K is one dimensional, and hence we can write the weight lattice in $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ as $\mathbb{Z}\delta + \mathbb{Z}\Lambda + \mathbb{Z}$. In this basis, the element ρ is given by $(0, -2, 1)$, and the simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are given by $(0, 0, 2)$ and $(1, 0, -2)$ resp.

Let u and v be formal variables representing the characters $e^{\hat{\alpha}_1}$ and $e^{\hat{\alpha}_2}$ resp. When we plug into the denominator formula, we get the Jacobi Triple product identity:

$$\sum_{m\in\mathbb{Z}}(-1)^m\, u^{m(m-1)/2}v^{m(m+1)/2} = \prod_{n\in\mathbb{N}}(1-u^{n+1}v^{n+1})(1-u^n v^{n+1})(1-u^{n+1}v^n)$$

An example: The level one Vacuum representation for $\mathbb{T}\ltimes\tilde{L}U(n)$ and $\mathbb{T}\ltimes\tilde{L}O(2n)$:
Consider the subspace of the (real) Hilbert space of trignometric functions Trig^n with values in the real $2n$-dimensional vector space underlying $\mathbb{C}^n$ with basis:

$$\{e_i\cos(k\theta),\ e_i\sin(s\theta),\quad k\geq 0,\ s>0,\ 1\leq i\leq 2n.\}$$

We endow Trig^n with an Euclidean inner product given by integrating the standard Euclidean inner product on $\mathbb{C}^n$:

$$\langle f(\theta), g(\theta)\rangle = \frac{1}{2\pi}\int \langle f(\theta), g(\theta)\rangle\, d\theta$$

One can now define C^*-algebra $\mathcal{C}$ generated by the Clifford relations in Trig^n:

$$f(\theta)\,g(\theta)+g(\theta)\,f(\theta) = \langle f(\theta), g(\theta)\rangle$$

Now notice that one has a canonical identification of $\mathrm{Trig}^n\otimes_{\mathbb{R}}\mathbb{C}$ as the completion of the Laurent polynomials on $\mathbb{C}^n\otimes_{\mathbb{R}}\mathbb{C}$:

$$\mathrm{Trig}^n\otimes_{\mathbb{R}}\mathbb{C} = L^2(S^1,\mathbb{C}^n\otimes_{\mathbb{R}}\mathbb{C}) = \mathbb{C}^n\hat{\otimes}_{\mathbb{R}}\mathbb{C}[z,z^{-1}],\quad z=e^{2\pi i\theta},$$

Furthermore, we may extend the Euclidean inner product on Trig^n complex linearly to a non-degenerate bilinear form on $\mathbb{C}^n\hat{\otimes}_{\mathbb{R}}\mathbb{C}[z,z^{-1}]$. Let J denote the complex structure on $\mathbb{C}^n$. Notice that the $\pm i$-eigenspaces of the complex linear extension of J yields an isotropic decomposition: $\mathbb{C}^n\otimes_{\mathbb{R}}\mathbb{C} = \overline{W}\oplus W$ [1]. This induces an isotropic decomposition of $\mathbb{C}^n\otimes_{\mathbb{R}}\mathbb{C}[z,z^{-1}] = H_+\oplus H_-$ with

$$H_+ = W[z]\oplus z\overline{W}[z],\quad H_- = \overline{W}[z^{-1}]\oplus z^{-1}W[z^{-1}]$$

We will denote by $\Lambda^*(H_+)$ the irreducible unitary representation of $\mathcal{C}$ given by the Hilbert completion of the exterior algebra on H_+ , with H_+ acting by exterior multiplication, and H_- acting by extending the contraction operator using the derivation property. By construction, $\mathbb{T}\ltimes LU(n)$ preserves the inner product on Trig^n, and hence it acts on $\mathcal{C}$ by algebra automorphisms. Since $\Lambda^*(H_+)$ is the unique representation of $\mathcal{C}$[2] , Schur's lemma says that we get a canonical projective action of $\mathbb{T}\ltimes LU(n)$ on $\Lambda^*(H_+)$ that intertwines the action of $\mathcal{C}$ twisted by the action of $\mathbb{T}\ltimes LU(n)$. The induced map $\mathbb{T}\ltimes LU(n)\to PU(\Lambda^*(H_+))$ lifts to the central extension giving rise to the universal central extension of $\mathbb{T}\ltimes LU(n)$. This representation is called the Fermionic Fock space. Notice this allows us to construct a non-trivial central extension of $\mathbb{T}\ltimes LK$ (and hence the universal one) of any compact Lie group K using an inclusion $K\subseteq U(n)$.

Note that the group $\mathbb{T}\ltimes LU(n)$ sits in an inclusion of groups:

$$\mathbb{T}\ltimes LU(n)\subset\mathbb{T}\ltimes LSO(2n)\subset\mathbb{T}\ltimes LO(2n).$$

[1] W is canonically isomorphic to $\mathbb{C}^n$ as a complex vector space

[2] Once we fix a polarization or equivalence class of maximal isotropic subspaces

The same argument we used for $\mathbb{T} \ltimes LU(n)$ shows that each of these groups act (projectively) on the Fermionic Fock space. Indeed, the representation is irreducible for each of these groups since it is irreducible for $\mathbb{T} \ltimes \tilde{L}U(n)$. Under the double cover of the identity component $\mathbb{T} \ltimes LSpin(2n) \longrightarrow \mathbb{T} \ltimes LSO(2n)$, the Fermionic Fock space decomposes into two irreducible summands (the even and odd exterior powers), see [PS] (Section 12.5). The odd and even representations are further decomposable representations for $\mathbb{T} \ltimes LSU(n)$ along the canonical lift of $\mathbb{T} \ltimes LSU(n)$ to $\mathbb{T} \ltimes LSpin(2n)$, see [PS] (Sections 10.6 and 13.1).

Section 12.4 [PS] describes an alternate geometric method to extend the projective representation $\Lambda^*(H_+)$ to $\mathbb{T} \ltimes LO(2n)$. One begins by studying what happens when we change the complex structure on the underlying real Hilbert space Trig^n. Let j be any other complex structure on the real Hilbert space Trig^n that is compatible with the inner product, and so that $[J, j]$ is a Hilbert-Schmidt operator. Then the complexification of j gives rise to another isotropic decomposition of the completed tensor product:

$$\mathbb{C}^n \hat{\otimes}_{\mathbb{R}} \mathbb{C}[z, z^{-1}] = H_+^j \oplus \overline{H}_+^j.$$

Taking the annhilator of the (super commtative) subalgebra of $\mathcal{C}$ generated by $\overline{H}_+^j$ and acting on $\Lambda^*(H_+)$ gives rise to a line (called Vacuum line). As such, one obtains a natural map:

$$\mathrm{Paff} : \mathcal{J}(\mathrm{Trig}^n) \longrightarrow \mathbb{P}(\Lambda^*(H_+)),$$

where $\mathcal{J}(\mathrm{Trig}^n)$ is the space of compatible complex structures j on Trig^n as above. The homotopy type of $\mathcal{J}(\mathrm{Trig}^n)$ is O/U and it admits a transitive action under the restricted Orthogonal group $O_{res}(\mathrm{Trig}^n)$. Under this action, the stabilizer of J is the (contractible) unitary group $U(L^2(S^1, \mathbb{C}^n))$, since Trig^n, endowed with the complex structure J is equivalent to $L^2(S^1, \mathbb{C}^n)$. Now, $\Lambda^*(H_+)$ can be identified with the completion of the space of holomorphic sections of the line bundle on $\mathcal{J}(\mathrm{Trig}^n)$ induced by Paff. The projective action of $\mathbb{T} \ltimes LO(2n)$ on $\Lambda^*(H_+)$ is induced by the translation action of $\mathbb{T} \ltimes LO(2n)$ on the base. Notice that the Loop groups $\mathbb{T} \ltimes LU(n)$ and $\mathbb{T} \ltimes LO(2n)$ are subgroups of $U_{res}(L^2(S^1, \mathbb{C}^n))$ and $O_{res}(\mathrm{Trig}^n)$ respectively and the natural inclusion $U_{res}(L^2(S^1, \mathbb{C}^n)) \subset O_{res}(\mathrm{Trig}^n)$ pulls Paff back to the universal central extension of $U_{res}(L^2(S^1, \mathbb{C}^n))$.

Let $x_1, x_2, \ldots, x_n$ denote the diagonal characters of the standard representation of $U(n)$ on $\mathbb{C}^n$, and let q denote the fundamental character of $\mathbb{T}$. Then observe that the character of the Fermionic Fock space is given by:

$$\mathrm{Ch}(\Lambda^*(H_+)) = \prod_{m=0}^{\infty} \prod_{i=1}^{n} (1 + x_i q^m)(1 + x_i^{-1} q^{m+1})$$

Remark 3.2. *The special case of the character formula when $n = 1$ is also interesting. If one expands the right hand side in terms of powers of x, then (using the Jacobi Triple product identity) each coefficient can be expressed as the character of a polynomial algebra. Indeed, this decomposition can be algebraically seen by decomposing the Fermionic Fock space under the subgroup of $\mathbb{T} \ltimes \tilde{L}U(1)$ consisting of loops of degree zero* [PS] *(Section 10.4). The expression of the Fermionic Fock space as an infinite sum of polynomial algebras is known as the Boson-Fermion correspondence.*

An example: The irreducible representation for $\mathbb{T} \ltimes \tilde{L}_{alg}K$ of lowest weight $-\rho$:

The denominator formula is a formal identity of characters and hence we may scale each character to get the identity:

$$\sum_{w \in \hat{W}} (-1)^w \, e^{w(-2\rho)} = e^{-2\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{2\alpha})^{\dim \, \hat{\mathfrak{g}}_\alpha}$$

Plugging this into the character formula gives us a formula for the representation $L_{-\rho}$:

$$\mathrm{Ch}(L_-\rho) = \frac{\sum_{w \in \hat{W}} (-1)^w \, e^{w(-2\rho)}}{e^{-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{\alpha})^{\dim \, \hat{\mathfrak{g}}_\alpha}} = \frac{e^{-2\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{2\alpha})^{\dim \, \hat{\mathfrak{g}}_\alpha}}{e^{-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{\alpha})^{\dim \, \hat{\mathfrak{g}}_\alpha}} = e^{-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 + e^{\alpha})^{\dim \, \hat{\mathfrak{g}}_\alpha}$$

This character formula suggests that as a (projective) representation of $\mathbb{T} \ltimes \tilde{L}_{alg}K$, we have:

$$L_{-\rho} = \Lambda^*(\hat{\eta}_+)$$

One way to construct this representation is via the idea introduced in the previous example. Consider the Lie algebra of $L_{alg}K$ with its canonical inner product. On complexifying this form, we get a $L_{alg}K$-invariant symmetric, non-degenerate, bilinear form on the Lie algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{h} \oplus \hat{\eta}_+ \oplus \hat{\eta}_-$. The sub spaces $\hat{\eta}_\pm$ are isotropic, dual subspaces which are orthogonal to $\mathfrak{h}$. It follows that the unique irreducible representation of the C^*-algebra generated by the corresponding Clifford algebra is given by: $\mathbb{S} := \mathbb{S}(\mathfrak{h}) \otimes \Lambda^*(\hat{\eta}_+)$, where $\mathbb{S}(\mathfrak{h})$ is the irreducible for the (finite dimensional) Clifford sub-algebra generated by $\mathfrak{h}$.

As in the previous example, Schur's lemma shows that $L_{alg}K$ acts on $\mathbb{S}$ by projective transformations that intertwine its action on the Clifford algebra. This representation can therefore be lifted to an honest representation of $\tilde{L}_{alg}K$. On calculating the character, it is clear that $\mathbb{S}$ decomposes into the representation $L_{-\rho}$, with multiplicity equal to the dimension of $\mathbb{S}(\mathfrak{h})$.

Fusion and the Verlinde ring, [A]:

Consider the category of positive energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of level l, with the property that the space of $\hat{\eta}_-$-invariants is finite dimensional (this space is called the space of lowest weight vectors). This is another way to describe the category of representations that are finite sums of irreducibles. Let A_l denote the Grothendieck group of this category. We have shown that this group is a finitely generated free abelian group with basis given by the weights in the Affine Alcove lA. Fusion is a very nice geometric construction of a commutative ring structure on A_l.

Conformal Blocks (Sketch):

Let $L_i, i = 1, 2, \ldots, k$ denote a collection of level l positive energy representations. Let Σ be a complex algebraic curve with k punctures labeled $x_i, i = 1, 2, \ldots, k$ and pick coordinates z_i with simple poles about these points. Consider the restriction homomorphism to the stalks:

$$r : \mathcal{O}(\Sigma/\{x_i\}, G) \longrightarrow \prod_i \mathcal{O}_{x_i}$$

The group on the right hand side admits a canonical central extension that extends the individual universal central extensions. However, the residue formula shows that this extension splits (canonically) when restricted along r. It can be shown that $\mathcal{O}(\Sigma/\{x_i\}, G)$

acts on $L^1_{alg} \otimes L^2_{alg} \otimes \ldots \otimes L^k_{alg}$ through the restriction to the punctures. Define the space $\mathbf{C}(\Sigma, \mathbf{x}, \mathbf{L})$ of conformal blocks to be the space dual to the coinvariants. In other words, we define:

$$\mathbf{C}(\Sigma, \mathbf{x}, \mathbf{L}) = \mathrm{Hom}(L^1_{alg} \otimes L^2_{alg} \otimes \ldots \otimes L^k_{alg}, \mathbb{C})^{\mathcal{O}(\Sigma/\{x_i\}, G)}$$

It can be shown [A] that this space is finite dimensional and the dimension is independent of choices. In fact, one can construct a vector bundle over the moduli space of punctured, uniformized curves of genus g, with the fiber over any curve being the space of conformal blocks. This bundle can be shown to have a projectively flat connection and it plays a very important part in conformal field theory.

Example 3.3. Consider the simplest example of $\Sigma = \mathbb{P}^1$ with one puncture x, labeled by representation L. Then $\mathcal{O}(\Sigma/x, G)$ is equivalent to $G(\mathbb{C}[z])$. This group contains the unipotent $\hat{U}_+$ as well as G as subgroups and hence the space of conformal blocks reduces to the space dual to G coinvariants of the space of zero-energy states V_0. Since V_0 is irreducible, this is non-zero if and only if V_0 is the trivial representation, or equivalently, if and only if L is the Vacuum representation [3].

Fusion:

Now assume L is an irreducible positive energy representation corresponding to an irreducible V of K. Define $\overline{L}$ to be the positive energy irreducible corresponding to the conjugate K representation $\overline{V}$. This correspondence $L \mapsto \overline{L}$ will be a duality in our category. Let Σ be a curve of genus zero with three punctures labeled x_1, x_2, x_3. Pick coordinates z_1, z_2, z_3 with simple poles about these points and consider three irreducible positive energy representations: $L^1, L^2, \overline{L}_3$. Define $C^{L_3}_{L_1, L_2}$ to be the dimension of the corresponding conformal block. Define a (bilinear) operation:

$$\mathrm{A}_l \otimes \mathrm{A}_l \longrightarrow \mathrm{A}_l, \qquad [L^1]\,[L^2] = \sum_{L_3} C^{L^3}_{L^1, L^2}\,[L^3]$$

where the sum on the right hand side runs over the (finite) set of irreducibles L^3. This bilinear operation is associative and commutative [A] making A_l into an (Verlinde) algebra. In fact, such operations can be defined for all genera and there is a beautiful theory of fusion rings. For example, one can use character theory over the complex numbers to derive beautiful formulas for these structure constants that involve cohomological information about the moduli of holomorphic bundles on riemann surfaces.

The map that sends an irreducible representation of K with lowest weight in $l\mathrm{A}$, to the corresponding irreducible positive energy representation, extends to a surjective ring homomorphism from the representation ring of K to A_l. Its kernel is generated by exactly those irreducible K representations V_λ with $\lambda \in C$, and satisfying $\langle \lambda - \rho_K, \alpha \rangle \in (l + \check{h})\mathbb{Z}$, for some root α of K, and where $\check{h}$ denotes the dual Coxeter number $\rho_K(h_{\alpha_0}) + 1$.

Example 3.4. The level l Verline algebra for $K = SU(2)$ is given by

$$\mathrm{A}_l = \mathbb{Z}[V]/\langle Sym^{l+1}[V] \rangle$$

where V denotes the fundamental representation of $SU(2)$ on $\mathbb{C}^2$.

[3] An easy generalization of this example using the Mittag-Leffler theorem, shows that the Vacuum represents the unit in the fusion ring described next.

4. Appendix: Three proofs, and a homotopy decomposition

Proof of Claim 1.1:

An element $A(z) \in L_{alg}G$ fixed under the involution $\hat{\sigma}$ is defined exactly by the property $\sigma\{A(\sigma_1(z))\} = A(z)$. Restricting to $S^1 \subset \mathbb{C}^*$ gives us the equality: $\sigma\{A(z)\} = A(z)$ on S^1, which is equivalent to saying that A maps S^1 to $K \subset G$. Now, given a unitary representation $K \to U(n)$, we may uniquely extend it to an algebraic map $G \to GL_n(\mathbb{C}) \subset \mathrm{End}_n(\mathbb{C})$. The element $A(z) \in G(\mathbb{C}[z, z^{-1}])$ maps to an element in $\mathrm{End}_n(\mathbb{C})[z, z^{-1}]$, and in particular, it has a finite fourier expansion.

Conversly, assume $B(z) \in L_{sm}K$ admits a finite fourier expansion under all unitary representations $K \to U(n) \subset \mathrm{End}_n(\mathbb{C})$. Pick a faithful representation that induces an algebraic map $G \subseteq \mathrm{End}_n(\mathbb{C})$. Then it is easy to see that $B(z)$ extends to a holomorphic function from $\mathbb{C}^*$ to $\mathrm{End}_n(\mathbb{C})$, which has a finite fourier expansion by assumption, and takes values in G [4]. Now on the other hand, $G(\mathbb{C}[z, z^{-1}])$ can be identified with exactly those holomorphic maps from $\mathbb{C}^*$ to $\mathrm{End}_n(\mathbb{C})$ that are in the image of G and are algebraic i.e admit finite fourier expansions in $\mathrm{End}_n(\mathbb{C})$. This implies in particular that $B(z)$ belongs to $G(\mathbb{C}[z, z^{-1}])$.

Proof of Claim 1.4: [PS](Sec 4.4).

To construct a topological group structure on $\tilde{L}K$, we proceed as follows. Recall the invariant two-form on $L\mathfrak{g}$ given by $\omega(\alpha, \beta) = Res\langle \alpha, d\beta\rangle$. Consider the quotient space:

$$\tilde{L}K = \{(\gamma,\, p,\, t) \in LK \times \Omega_0(\mathfrak{g}) \times S^1\} \,/ \sim$$

where $\Omega_0(\mathfrak{g})$ is the space of based loops in $L\mathfrak{g}$, and the equivalence relation is given by:

$$(\gamma,\, p,\, t) \sim (\gamma,\, q,\, t\, exp(\int_{D(p*\overline{q})} \omega))$$

where $D(p * \overline{q})$ is any disc in $L\mathfrak{g}$ with boundary given by $p * \overline{q}$. This space admits a well-defined topological group structure given by:

$$(\gamma,\, p,\, t)(\lambda,\, q,\, s) = (\gamma\lambda,\;\; p * \gamma(q),\;\; st).$$

Under this group structure, $\tilde{L}K$ is a central extension of LK by S^1. Restricting the group extension to the based loops ΩK, we get a principal extension of ΩK by S^1, which is classified by $\mathbb{Z} \cong H^2(\Omega K, \mathbb{Z}) = H^3(K, \mathbb{Z}) = \mathrm{Hom}(\pi_3(K), \mathbb{Z})$. Under these sequence of identifications, the generator of $\mathbb{Z}$ corresponds to the homomorphism $\sigma \mapsto 2\pi i \int_\sigma \omega$, where σ is seen as a map from S^2 to LK. This generator is exactly the one used to define the extension above. Hence we recover the universal central extension.

Proof of Theorem 1.2:

Closely related to the action of the affine Weyl group $\hat{W}$ on the affine subspace in $\hat{\mathfrak{h}}^*_{\mathbb{R}}/\langle\delta\rangle$ (see end of Lecture II) is the following dual action of $L_{alg}K$ on an affine subspace $\mathcal{A}$ in $i\hat{\mathfrak{k}}/\mathbb{R}\langle k\rangle$:

$$\mathcal{A} = \{h(z) \in i\hat{\mathfrak{k}}/\mathbb{R}\langle k\rangle \,|\, \delta(h(z)) = 1\,\}, \quad \text{with} \quad g(z)(d + f(z)) = d + g(z)f(z) + z\, g'(z)\, g(z)^{-1}$$

where $\hat{\mathfrak{k}}$ denotes the Lie algebra of $\mathbb{T} \ltimes L_{alg}K$, $f(z) \in \mathfrak{g}[z, z^{-1}]$ and $g(z) \in L_{alg}K$.

[4] To see this, assume that G is defined by algebraic equations

One can think of this as the action of the (polynomial) gauge group on the space of (polynomial) connections on the trivial K bundle over S^1 and consequently, that the subgroup of pointed loops $\Omega_{alg}K$ acts freely on $\mathcal{A}$. It is easy to see that the quotient space is equivalent to K (via the holonomy map). This construction shows that $\Omega_{alg}K$ is homotopy equivalent to the space of continuous loops ΩK, thus proving Theorem 1.2 (in lecture I).

A homotopy decomposition of the classifying space $B\tilde{L}_{alg}K$: [NK].

Recall the space $\mathcal{A}$ of polynomial connections above. This space contains the (dual) alcove $\check{A} = \{h \in d \oplus \mathfrak{h}_{\mathbb{R}} \,|\, \hat{\alpha}_i(h) \leq 0 \quad i \leq n+1\}$ as a subspace. It can be shown [NK] that $\check{A}$ is a fundamental domain for the action of $\tilde{L}_{alg}K$ action on $\mathcal{A}$ and hence the following canonical equivariant map is surjective:

$$\tilde{L}_{alg}K \times \check{A} \longrightarrow \mathcal{A}$$

One may index the interiors of the faces of the simplex $\check{A}$ by subsets $I \subset \{1, 2, \ldots, n+1\}$ of positive roots whose vanishing set describes that face. All proper subsets of the set $\{1, 2, \ldots, n+1\}$ correspond to faces, with the empty set corresponding to the interior of $\check{A}$. This collection of subsets has a natural poset structure under inclusion, and we call the poset $\mathcal{C}$. Let K_I denote the (compact Lie) pointwise isotropy group at (any) interior point on the face $\check{A}_I$ corresponding to I. For example, corresponding to the empty subset $\varnothing \in \mathcal{C}$, the isotropy group$K_{\varnothing}$ of the interior points of $\check{A}$ is the maximal torus $S^1 \times T$. Hence we get an equivariant homeomorphism:

$$\mathcal{A} \cong (\tilde{L}_{alg}K \times \check{A})/\sim \qquad (g, x) \sim (h, y) \quad \text{if} \quad x = y \in \check{A}_I, \quad gK_I = hK_I.$$

We may rephrase the above observation in terms of homotopy theory by saying that $\mathcal{A}$ can be described as the homotopy colimit of the homogeneous spaces $\tilde{L}K/K_I$ along the poset $\mathcal{C}$. The following result is obtained by taking the Borel construction of the contractible space $\mathcal{A}$:

Theorem 4.1. *The classifying space $B\tilde{L}_{alg}K$ of the group $\tilde{L}_{alg}K$ is homotopy equivalent to the homotopy colimit of the classifying spaces BK_I of compact Lie groups, along the poset $\mathcal{C}$.*

This theorem allows us to set up a Mayer-Vetoris type spectral sequence to make various homological computations with $B\tilde{L}_{alg}K$. See [NK] for more details.

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