Lectures on Loop groups

IST Portugal, June, 2011

The Basic Definitions:

- We set up notation by letting K denote a compact, simply connected, simple Lie group. By G we shall mean the complexification of K. So, up to isomorphism, G is the unique simple complex algebraic group which includes K as a maximal compact subgroup.
- There is an anti-linear involution σ on G called the Cartan involution whose fixed points is K.
- For example if we take $K = SU(n)$, then $G = SL_n(\mathbb{C})$ and σ is given by $\sigma(A) = (A^*)^{-1}$.

Two types of Loop groups

- We define two types of loop groups $L_{sm}K$ and $L_{alg}K$. The group $L_{sm}K$ is the (pointwise) group of smooth maps from S^1 to K in the compact open topology.
- The group $L_{alg}K$ may be defined as the subgroup of maps with finite Fourier expansions under all unitary representations of K.
- However, the topology on $L_{alg}K$ is given by the direct limit topology induced by the compact sub-spaces of $L_{sm}K$ that consist of maps of increasing degree under some faithful unitary representation of K.

- **Theorem**

  The map $L_{alg}K \to L_{sm}K$ is a homotopy equivalence. Hence the homotopy/homology of the Loop group is independent of the type of loops.

Algebraic Loops as fixed points:

- Alternatively, to define $L_{alg}K$, we may mimic the above story by starting with the complex form $L_{alg}G = G(\mathbb{C}[z, z^{-1}])$ of $\mathbb{C}^*$-valued points in G.

  Then define $L_{alg}K$ as the fixed points under the (loop) Cartan involution $\hat{\sigma}(A)(z) = \sigma\{A(\sigma_1(z))\}$, where σ_1 is the automorphism of $\mathbb{C}^*$ sending z to $\overline{z}^{-1}$.

- Claim

  Let $\hat{\sigma}$ denote the Cartan involution on $L_{alg}G$, then the set of fixed points of $\hat{\sigma}$ is exactly the set of maps $A(z) \in L_{alg}G$ such that $A(S^1) \subseteq K$. In particular, seen as an element in $L_{sm}K$, the map $A(z)$ has a finite fourier expansion under any unitary representation of K. Conversely, any map $A(z) \in L_{sm}K$ that has a finite fourier expansion under all unitary representations of K, extends uniquely to an element in $L_{alg}G$.

The central extension, the extended Loop group and the Affine Weyl group:

- Since the algebraic loops behave essentially the same as the smooth loops, and so we ignore the subscript unless it is necessary.
- Notice that the map $LK \to K$ given by evaluating at $1 \in S^1$ is split by the constant loops, hence $LK = K \ltimes \Omega K$. It follows that $H^2(LK, \mathbb{Z}) = H^2(\Omega K, \mathbb{Z}) = \mathbb{Z}$.
- Let $\tilde{L}K$ denote the principal S^1 bundle over LK representing the generator of $H^2(LK, \mathbb{Z})$.

- Claim

  The space $\tilde{L}K$ has the structure of a universal S^1-central extension of LK, called the Affine group. In particular, $\tilde{L}K$ is 2-connected and any nontrivial S^1-central extension of LK is a finite quotient of $\tilde{L}K$.

  Let $\mathbb{T}$ denote the circle acting on LK by rotation: $e^{i\theta}.A(z) = A(e^{i\theta}z)$. Then the $\mathbb{T}$ action on LK lifts to $\tilde{L}K$, and the group $\mathbb{T} \ltimes \tilde{L}K$ is called the extended Affine group.

The Weyl group and the extended Affine Lie algebra

- Let T denote the maximal torus of K, and let $W = N_K(T)/T$ be the finite Weyl group. Then $\mathbb{T} \times S^1 \times T \subset \mathbb{T} \ltimes \tilde{L}K$ is a maximal torus, and its Weyl group $\tilde{W}$ is called the Affine Weyl group. This group is isomorphic to $W \ltimes \pi_1(T)$.
- Let $\mathfrak{g}$ denote the (complex) Lie algebra of G. Then the Lie algebra of $L_{sm}G$ can easily be seen to be the space of smooth maps form $\mathbb{C}^*$ to $\mathfrak{g}$. In the case of algebraic loops, one formally defines the Lie algebra of $L_{alg}G$ to be $L_{alg}\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$. Define the extended Affine Lie algebra $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}} = \mathbb{C}\langle d\rangle \oplus \mathbb{C}\langle k\rangle \oplus \mathfrak{g}[z, z^{-1}], \quad [d, \alpha z^n] = n\alpha z^n$$

$$[\alpha z^n, \beta z^m] = [\alpha, \beta] z^{m+n} + k\langle \alpha, \beta\rangle Res(z^n d(z^m)),$$

with $\langle\ ,\ \rangle$ denoting the Cartan-Killing form on $\mathfrak{g}$, $\mathbb{C}\langle d\rangle$ denoting the complexified Lie algebra of $\mathbb{T}$, and $\mathbb{C}\langle k\rangle$ denoting the complexified Lie algebra of the center S^1.

Representation Theory:

- Given a unitary representation of $\mathbb{T} \ltimes \tilde{L}K$ in a Hilbert space $\mathcal{H}$, we may restrict to $\mathbb{T}$ to obtain a dense inclusion:

$$\bigoplus q^n \mathcal{H}_n \subseteq \mathcal{H}$$

  where q is the fundamental character of $\mathbb{T}$, and $\mathcal{H}_n$ is the corresponding isotypical summand. We say $\mathcal{H}$ is a representation of positive (resp. negative) energy if each $\mathcal{H}_n$ is finite dimensional and $\mathcal{H}_n = 0$ for $n < 0$ (resp. $n > 0$). Note that if $\mathcal{H}_n = 0$ for sufficiently small (resp. large) values of n, we may tensor $\mathcal{H}$ by a suitable character of $\mathbb{T}$ to make it positive (resp. negative) energy. The category of unitary positive energy (resp. negative energy) representations of $\mathbb{T} \ltimes \tilde{L}K$ is semisimple.
- If $\mathcal{H}$ is an irreducible, then by Schur's lemma, the central circle acts by a fixed character μ on $\mathcal{H}$. The character μ is called the level of $\mathcal{H}$. There are finitely many irreducible positive (resp. negative) energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of a fixed level.

Character Formula and Fusion

There is a nice (Kac-Weyl) character formula for the character of any irreducible positive/negative energy representation when restricted to $\mathbb{T} \times S^1 \times T$. Special cases of this formula are several well known identities known as Macdonald's Identities.

Now let $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ denote two irreducible positive energy representations of respective levels λ and μ. Then the (vector space) tensor product $\mathcal{H}_\lambda \otimes \mathcal{H}_\mu$ can be decomposed into an infinite sum of positive energy representations of level $\lambda + \mu$. Hence, the naive tensor product does not preserve the level. Now let A_λ denote the Grothendieck group of the category of positive energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of a fixed level λ and finite composition series. Then A_λ is a finitely generated free abelian group that supports a (subtle) commutative product called the fusion product. The induced ring structure on A_λ is called the fusion ring or Verlinde ring. The unit in this ring is called the Vacuum representation of level λ.

Example

The Hilbert completion of the exterior algebra $\Lambda^*(\mathbb{C}^n[z]) \otimes \Lambda^*(z\overline{\mathbb{C}}^n[z])$ is an irreducible positive energy representation of $\mathbb{T} \ltimes \tilde{L}U(n)$ of level one, that contains all level one representations of $\mathbb{T} \ltimes \tilde{L}SU(n)$. This is also known as the Fermionic Fock space representation.

Remark

Even the case $n = 1$ is interesting: When one decomposes the Fermionic Fock space as a representation of degree zero smooth maps: $\mathbb{T} \ltimes \tilde{L}_0U(1)$, one gets an infinite sum of (Bosonic) polynomial algebras. This identity of representations is known as the Boson-Fermion correspondence.

Structure theory of the Lie algebra

First we review some finite dimensional theory:

Recall that the finite dimensional Lie algebra $\mathfrak{g}$ admits a symmetric, bilinear, non-degenerate, $\mathfrak{g}$-invariant (Cartan-Killing) form given by: $\langle \alpha, \beta \rangle = \mathrm{Tr}\, ad(\alpha)\, ad(\beta)$.

Let $\mathfrak{h}$ be the (Cartan) maximal abelian sub-algebra of $\mathfrak{g}$ given by the complexification of the Lie algebra of T. The above form restricts to a non-degenerate form on $\mathfrak{h}$.

Under the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$, one has the eigenspace decomposition called triangular decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \eta_+ \oplus \eta_-, \quad \eta_\pm = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}$$

where $\Delta_+ \subset \mathfrak{h}^*$ is the set of (non-zero) positive roots defined by the property $[h, x] = \pm\alpha(h)x$ for $h \in \mathfrak{h}, x \in \mathfrak{g}_{\pm\alpha}$. The spaces $\mathfrak{g}_{\pm\alpha}$ are called the root spaces of $\mathfrak{g}$. The root spaces $\mathfrak{g}_{\pm\alpha}$ are generated by elements $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$ called a root vectors. The co-roots: $h_\alpha = [e_\alpha, f_\alpha] = 2\alpha^*/\langle \alpha, \alpha \rangle$, span $\mathfrak{h}$.

Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ be a real form (so that each α takes real values on $\mathfrak{h}_{\mathbb{R}}$). Then $h_\alpha \in \mathfrak{h}_{\mathbb{R}}$. We define the anti-dominant Weyl chamber of $\mathfrak{h}^*$ to be the polyhedral cone:

$$C = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid \beta(h_\alpha) \leq 0, \forall\, \alpha \in \Delta_+\}$$

There exist a set of positive roots $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta_+$ (called the simple roots) such they are linearly independent in $\mathfrak{h}_{\mathbb{R}}^*$ and such that any $\alpha \in \Delta_+$ is a non-negative integral linear combination of the simple roots. There is a unique positive root $\alpha_0 \in \Delta_+$ (called the highest root) which is the largest in the sense that $\alpha_0 - \alpha$ is a non-negative integral combination of the simple roots for any $\alpha \in \Delta_+$. By convention, we normalize the Cartan-Killing form so that $\langle \alpha_0, \alpha_0 \rangle = 2$.

The Weyl group is generated by the set $\{r_{\alpha_1}, r_{\alpha_2}, \ldots, r_{\alpha_n}\}$ of reflections about the walls of the Weyl chamber given by the planes $\beta(h_{\alpha_i}) = 0$. The explicit formula for the action of r_{α_i} on $\beta \in \mathfrak{h}_{\mathbb{R}}^*$ is given by $r_{\alpha_i}(\beta) = \beta - \beta(h_{\alpha_i})\, \alpha_i$.

Example

Consider the example of $\mathfrak{g} = sl_3(\mathbb{C})$ of traceless 3×3 complex matrices. Then $\mathfrak{h}$ is given by the diagonal matrices and $\eta_\pm$ is the space of upper-triangular (resp. lower-triangular) matrices. The positive roots are given by $\alpha_1, \alpha_2, \alpha_0$ with $\alpha_0 = \alpha_1 + \alpha_2$ defined as follows: For a diagonal matrix h with entries (x_1, x_2, x_3) we have:

$$\alpha_1(h) = x_1 - x_2, \quad \alpha_2(h) = x_2 - x_3, \quad \alpha_0(h) = x_1 - x_3.$$

The classes h_α are given by $h_{\alpha_1} = (1, -1, 0)$, $h_{\alpha_2} = (0, 1, -1)$ and $h_{\alpha_0} = (1, 0, -1)$ in the diagonal representation. In addition:

$$e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, e_{\alpha_0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f_\alpha = e_\alpha^T$$

The Weyl group is isomorphic to the symmetric group Σ_3 generated by two reflections about the walls of C (that subtend an angle of $\pi/3$).

The Affine Lie algebra:

Recall that the Affine Lie algebra $\hat{\mathfrak{g}} = \mathbb{C}\langle d \rangle \oplus \mathbb{C}\langle k \rangle \oplus \mathfrak{g}[z, z^{-1}]$:

$$[f, g](z) = [f(z), g(z)] + k\, Res\langle f, dg \rangle, \quad [d, f] = z \frac{\partial f}{\partial z}$$

The cartan involution on $\hat{\mathfrak{g}}$ is given by:

$$\hat{\sigma}(\alpha z^n) = \sigma(\alpha) z^{-n}, \quad \hat{\sigma}(k) = -k, \quad \hat{\sigma}(d) = -d.$$

Let $\hat{\mathfrak{k}}$ denote the fixed points of $\hat{\sigma}$. We formally identify it with the Lie algebra of $\mathbb{T} \ltimes \tilde{L}_{alg} K$.

Identifying $\mathfrak{h}$ with $\mathfrak{h} \otimes 1 \subset \hat{\mathfrak{g}}$, we have the Cartan sub-algebra:

$$\hat{\mathfrak{h}} = \mathbb{C}\langle d \rangle \oplus \mathbb{C}\langle k \rangle \oplus \mathfrak{h}, \quad \hat{\mathfrak{h}}_{\mathbb{R}} = \mathbb{R}\langle d \rangle \oplus \mathbb{R}\langle k \rangle \oplus \mathfrak{h}_{\mathbb{R}}, \quad \hat{\mathfrak{h}}^*_{\mathbb{R}} = \mathbb{R}\langle \delta \rangle \oplus \mathbb{R}\langle \Lambda \rangle \oplus \mathfrak{h}^*_{\mathbb{R}}$$

where $\delta, \Lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}}$ denote the elements dual to d and k under the above decomposition. The triangular decomposition is given by $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \hat{\eta}_+ \oplus \hat{\eta}_-$ with

$$\begin{aligned} \hat{\eta}_+ = z\,\mathfrak{g}[z] \oplus \eta_+ &= \{f(z) \in \mathfrak{g}[z] \,|\, f(0) \in \eta_+\} \\ \hat{\eta}_- = z^{-1}\mathfrak{g}[z^{-1}] \oplus \eta_- &= \{f(z) \in \mathfrak{g}[z^{-1}] \,|\, f(\infty) \in \eta_-\} \end{aligned}$$

We may describe the positive roots of the $\hat{\mathfrak{h}}$-action on $\hat{\eta}_+$:

$$\hat{\Delta}_+ = \{(n\delta + \epsilon\alpha) \in \hat{\mathfrak{h}}^*_{\mathbb{R}} \,|\, n \in \mathbb{N}, \alpha \in \Delta_+\}$$

with $n > 0$ and $\epsilon \in \{-1, 0, 1\}$, or with $n = 0$, and $\epsilon = 1$.
The individual root spaces are given by:

$$\hat{\mathfrak{g}}_\alpha = \mathbb{C}\langle e_\alpha \rangle, \quad \hat{\mathfrak{g}}_{n\delta+\alpha} = \mathbb{C}\langle e_\alpha\, z^n \rangle, \quad \hat{\mathfrak{g}}_{n\delta-\alpha} = \mathbb{C}\langle f_\alpha\, z^n \rangle, \quad \hat{\mathfrak{g}}_{n\delta} = \mathfrak{h}\, z^n$$

The roots $\{n\delta\}$ are called the imaginary roots and are the only positive roots with multiplicity bigger than one.

Notice that we have a collection of simple roots given by:

$$\{\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_{n+1}\} = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \delta - \alpha_0\} \quad \text{with}$$

$$e_{\hat{\alpha}_i} = e_{\alpha_i}, \quad f_{\hat{\alpha}_i} = f_{\alpha_i} \quad \text{for} \quad i \leq n,$$

$$e_{\hat{\alpha}_{n+1}} = f_{\alpha_0}\, z, \quad f_{\hat{\alpha}_{n+1}} = e_{\alpha_0}\, z^{-1}$$

The simple co-roots $h_{\hat{\alpha}_i} = [e_{\hat{\alpha}_i}, f_{\hat{\alpha}_i}]$ are:

$$h_{\hat{\alpha}_i} = h_{\alpha_i} \quad \text{for} \quad i \leq n, \quad \text{and} \quad h_{\hat{\alpha}_{n+1}} = -(k + h_{\alpha_0})$$

Some corresponding subgroups of the Loop group

For the algebraic group G, the subgroups corresponding to the subalgebras $\mathfrak{h}, \eta_+, \eta_-$ are H, U_+, U_-, where H is called the Cartan subgroup (it is the complexification of the maximal torus of K), and $U_\pm$ are called the positive (resp. negative) unipotent subgroups. In the example of $SL_n(\mathbb{C})$, the group H is the diagonal matrices, and $U_\pm$ are the upper (resp. lower) triangular matrices.

There are corresponding subgroups of the complex form $\mathbb{C}^* \ltimes \tilde{G}(\mathbb{C}[z, z^{-1}])$ that correspond to $\hat{\mathfrak{h}}, \hat{\eta}_+, \hat{\eta}_-$ given by $\hat{H}, \hat{U}_+, \hat{U}_-$ resp., with $\hat{H} = \mathbb{C}^*(d) \times \mathbb{C}^*(k) \times H$, and $\hat{U}_\pm$ are:

$$\{f(z) \in G(\mathbb{C}[z]) \,|\, f(0) \in U_+\}, \quad \{f(z) \in G(\mathbb{C}[z^{-1}]) \,|\, f(\infty) \in U_-\}$$

We may define the Borel sub-algebras by $\hat{b}_\pm = \hat{\mathfrak{h}} \oplus \hat{\eta}_\pm$. The corresponding groups generated by $\hat{H}$ and $\hat{U}_\pm$ are called the Borel subgroups $\hat{B}_\pm$.

The Affine Weyl group

On $\hat{\mathfrak{h}}_{\mathbb{R}}$ there is a non-degenerate, symmetric form (which is not positive definite) given by extending the Cartan-Killing form on $\mathfrak{h}_{\mathbb{R}}$, demanding that $\mathbb{R}\langle d\rangle \oplus \mathbb{R}\langle k\rangle$ be orthogonal to $\mathfrak{h}_{\mathbb{R}}$ and the formulas: $\langle d, k\rangle = 1$, and $\langle d, d\rangle = \langle k, k\rangle = 0$. This form induces a dual form on $\hat{\mathfrak{h}}_{\mathbb{R}}^*$ and the Affine Weyl group $\hat{W}$ is generated by reflections $r_{\hat{\alpha}_i}$ about the walls $\beta(h_{\hat{\alpha}_i}) = 0$ of the anti-dominant Affine Weyl chamber $\hat{C}$ in $\hat{\mathfrak{h}}_{\mathbb{R}}^*$. We have the following formulas:

$$r_{\hat{\alpha}_i}(\alpha) = r_{\alpha_i}(\alpha) \quad i \leq n, \qquad r_{\hat{\alpha}_{n+1}}(\alpha) = r_{\alpha_0}(\alpha) + \alpha(h_{\alpha_0})\,\delta$$

$$r_{\hat{\alpha}_i}(\Lambda) = \Lambda \quad i \leq n, \qquad r_{\hat{\alpha}_{n+1}}(\Lambda) = \Lambda + \delta - \alpha_0$$

$$r_{\hat{\alpha}_i}(\delta) = \delta \qquad i \leq n+1$$

We observe two facts. Firstly, one observes that $\hat{W}$ preserves the affine slices (called levels) given by vectors of the form $l\Lambda \oplus \mathbb{R}\langle\delta\rangle \oplus \mathfrak{h}_{\mathbb{R}}^*$. Secondly, the action of $\hat{W}$ is well-defined and faithful on the vector space $\hat{\mathfrak{h}}_{\mathbb{R}}^*/\mathbb{R}\langle\delta\rangle$. Working modulo δ, we have the following formulas:

$$r_{\hat{\alpha}_{n+1}} r_{\alpha_0}(\Lambda) = \Lambda - \alpha_0, \quad r_{\hat{\alpha}_{n+1}} r_{\alpha_0}(\alpha) = \alpha$$

It follows that the normal subgroup of $\hat{W}$ generated by $r_{\hat{\alpha}_{n+1}} r_{\alpha_0}$ is isomorphic to the lattice I in $\mathfrak{h}_{\mathbb{R}}^*$ generated by the W orbit of α_0. From this observation we see that

$$\hat{W} = \langle r_{\hat{\alpha}_1}, r_{\hat{\alpha}_2}, \ldots, r_{\hat{\alpha}_n}, r_{\hat{\alpha}_{n+1}} \rangle = \langle r_{\hat{\alpha}_1}, r_{\hat{\alpha}_2}, \ldots, r_{\hat{\alpha}_n}, r_{\hat{\alpha}_{n+1}} r_{\alpha_0} \rangle \cong W \ltimes \mathrm{I}$$

The Affine Alcove

Recall the action of the Affine Weyl group on the affine slices (or levels) described above: $l\Lambda \oplus \mathfrak{h}_{\mathbb{R}}^*$ where we are working modulo δ. Under this action, the subgroup I acts by translation by l I (l times I), and W acts through its action on $\mathfrak{h}_{\mathbb{R}}^*$.

It is easy to see that the fundamental domain of this action is contained in the anti-dominant Affine Weyl chamber $\hat{C}$, and by projecting to $\mathfrak{h}_{\mathbb{R}}^*$ it can be identified with the (scaled) anti-dominant Affine Alcove lA, where

$$l\mathrm{A} = \{\beta \in C \mid \beta(h_{\alpha_0}) \geq -l\}$$

It is important to take note that lA is non-empty if and only if $l \geq 0$.

The space of polynomial connections

Closely related to the above action of $\hat{W}$ on the affine subspace in $\hat{\mathfrak{h}}^*_{\mathbb{R}}/\langle\delta\rangle$ is the following action of $L_{alg}K$ on an affine subspace $\mathcal{A}$ in $i\hat{\mathfrak{k}}/\mathbb{R}\langle k\rangle$ given by: $\mathcal{A} = \{h(z) \in i\hat{\mathfrak{k}}/\mathbb{R}\langle k\rangle \,|\, \delta(h(z)) = 1\}$.

$$g(z)(d + f(z)) = d \;+\; g(z)f(z) + z\,g(z)^{-1}g'(z)$$

where $f(z) \in \mathfrak{g}[z, z^{-1}]$ and $g(z) \in L_{alg}K$.

One can think of this as the action of the (polynomial) gauge group on the space of (polynomial) connections on the trivial K bundle over S^1 and consequently, that the subgroup of pointed loops $\Omega_{alg}K$ acts freely on $\mathcal{A}$. It is easy to see that the quotient space is equivalent to K (via the holonomy map). This construction shows that $\Omega_{alg}K$ is homotopy equivalent to the space of continuous loops ΩK.

The theory of Positive energy Representations

Let $\mathcal{H}$ be a level l, positive energy representation of $\mathbb{T} \ltimes \tilde{L}K$. We decompose it under the compact subgroup $\mathbb{T} \times S^1 \times K$ to get a dense subspace of $\mathcal{H}$:

$$\mathcal{H}_{alg} = \bigoplus q^n V_n$$

where V_n are finite dimensional representations of $S^1 \times K$ of level l. By assumption $V_n = 0$ for $n < 0$. The subspace V_0 is called the states of zero energy, and we assume that $V_0 \neq 0$. By the assumption of positive energy, all the negative root vectors of $\hat{\mathfrak{g}}$ not in $\mathfrak{g}$ act trivially on V_0. It follows that V_0 is a representation of the sub algebra $\hat{b}_- + \mathfrak{g}$, with the action factoring through the reductive sub-algebra $\hat{\mathfrak{h}} + \mathfrak{g}$. So we get:

$$\pi : \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{b}_- + \mathfrak{g})} V_0 \longrightarrow \mathcal{H}_{alg}$$

By the triangular decomposition and the PBW theorem, we observe that $\mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{b}_- + \mathfrak{g})} V_0$ is isomorphic as an $\hat{\mathfrak{h}}$-module to $\mathcal{U}(\hat{\mu}_+) \otimes V_0$, where $\hat{\mu}_+ \subset \hat{\eta}_+$ is the sub-algebra generated by all positive root vectors not in $\mathfrak{g}$.

Now let us decompose V_0 into a sum of irreducibles with multiplicity: $V_0 = \oplus V_{\tau_i}$, where V_{τ_i} is an irreducible with lowest weight $\tau_i \in \hat{\mathfrak{h}}_{\mathbb{R}}^*$ of the form $l\Lambda + \lambda_i$ with λ_i a weight in the anti-dominant Weyl chamber C of $\mathfrak{g}$. In addition, since $\mathcal{H}$ has an action of the Affine Weyl group, we observe that $r_{\hat{\alpha}_{n+1}}$ acts on the weight $l\Lambda + \lambda_i$ to yield another weight in $\mathcal{H}_{alg}$. It follows from the explicit formulas that $\lambda_i(h_{\alpha_0}) + l \geq 0$ and hence that λ_i belongs to the anti-dominant Affine Alcove lA.

Now assume that $\mathcal{H}$ is irreducible. It follows that the representation $\mathcal{H}_{alg}$ of the Affine Lie algebra $\hat{\mathfrak{g}}$ is also irreducible and that π is surjective. In addition, we deduce that V_0 must be an irreducible representation of $S^1 \times K$ (since the $\hat{\mathfrak{g}}$ span on V_0 cannot have other vectors of zero-energy). Hence $V_0 = V_\tau$ for some weight $\tau \in \hat{\mathfrak{h}}_{\mathbb{R}}^*$ of the form $l\Lambda + \lambda$ with λ belonging to the anti-dominant Affine Alcove lA. It is easy to see that the positive energy $\hat{\mathfrak{g}}$-module of the form $\mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{b}_- + \mathfrak{g})} V_\tau$ has a unique maximal ideal, and so the corresponding irreducible quotient L_τ is an irreducible representation isomorphic to $\mathcal{H}_{alg}$.

L_τ as a dense subspace of holomorphic sections:

Let $\hat{G}$ denote $\mathbb{C}^* \ltimes \tilde{G}(\mathbb{C}[z, z^{-1}])$. Given $\tau \in lA$, consider the holomorphic line bundle over the (complex) homogeneous space $\hat{G}/\hat{B}_+$ given by $\mathcal{L}_\tau = \hat{G} \times_{\hat{B}_+} \mathbb{C}_\tau$, with $\hat{B}_+$ acting on $\mathbb{C}$ via the character τ as described earlier. We claim that the space of holomorphic sections of $\mathcal{L}_\tau$ contains a dense subspace isomorphic to L_τ. To see this, it is sufficient to show that the space of $\hat{\eta}_-$ invariant vectors is one dimensional, since these invariant vectors index exactly the irreducible lowest energy states inside any (sub) module. We establish this by restricting the bundle to the contractible dense open set equivalent to $\hat{U}_-$ (whose lie algebra is $\hat{\eta}_-$). By equivariance, we see that the only such sections are the constants. In addition, if τ belongs to lA, this constant section can be shown to extend to a global section.

Theorem

Taking lowest energy states establishes a bijection between irreducible positive energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of level l, and irreducible representations of K with lowest weight in lA.

Complete reducibility

Now returning to an arbitrary positive energy representation $\mathcal{H}$, with zero energy states given by V_0, let $v \in V_0$ be a lowest weight vector with anti-dominant weight τ. Let $p : \mathcal{H}_{alg} \to \mathbb{C}\langle v \rangle$ denote a $\mathbb{T} \times S^1 \times T$-equivariant projection. This extends to a $\mathbb{T} \ltimes \tilde{L}K$-equivariant map:

$$\mathcal{H}_{alg} \longrightarrow \Omega_{hol}(\mathcal{L}_\tau), \qquad \xi \mapsto \ f(z) \mapsto p(f(z)^{-1}\xi)$$

where we have identified $\Omega_{hol}(\mathcal{L}_\tau)$ with a suitable completion of the space of (right) $\hat{B}_+$-equivariant polynomial maps from $\hat{G}$ to $\mathbb{C}\langle v \rangle$. Splitting the map given above allows us to prove complete reducibility.

The character formula

The interpretation of the irreducible L_τ as a subspace of holomorphic sections allows us to calculate its character (modulo technical details) via the fixed point formula:

$$\mathrm{Ch}(L_\tau) = \frac{\sum_{w\in\hat{W}} (-1)^w \, e^{w(\tau-\rho)}}{e^{-\rho} \prod_{\hat{\alpha}\in\hat{\Delta}_+} (1-e^\alpha)} = \sum_{w\in\hat{W}} \frac{(-1)^w \, e^{w(\tau)}}{e^{w(\rho)-\rho} \prod_{\hat{\alpha}\in\hat{\Delta}_+} (1-e^\alpha)}$$

where $\rho \in \hat{\mathfrak{h}}^*_{\mathbb{R}}$ is (uniquely) defined by $\rho(h_{\hat{\alpha}_i}) = 1$ for all $i \leq n+1$. Here, one should think of the expression $(-1)^w e^{w(\rho)-\rho} \prod_{\hat{\alpha}\in\hat{\Delta}_+} (1-e^\alpha)$ as the character of the complex Spinor representation for the complex Clifford algebra on the tangent space at the $\mathbb{T} \times S^1 \times T$ fixed point given by $w\hat{B}_+ \in \hat{G}/\hat{B}_+$.

Notice in particular that for the trivial level 0 representation, one has the Weyl-Kac denominator formula:

$$\sum_{w\in\hat{W}} (-1)^w \, e^{\rho - w(\rho)} = \prod_{\hat{\alpha}\in\hat{\Delta}_+} (1-e^\alpha)$$

The denominator formula for $\mathbb{T} \ltimes \tilde{L}SU(2)$:

Consider the case of the group $K = SU(2)$. Let $\hat{\mathfrak{g}}$ denote the Affine Lie algebra with two simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$. The positive roots are given by:

$$\hat{\Delta}_+ = \{n\delta + \hat{\alpha}_1,\ n\delta + \hat{\alpha}_2; \quad n \geq 0\} \coprod \{n\delta; \quad n > 0\}, \text{ where } \delta = \hat{\alpha}_1 + \hat{\alpha}_2.$$

Here the maximal torus of K is one dimensional, and hence we can write the weight lattice in $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ as $\mathbb{Z}\delta + \mathbb{Z}\Lambda + \mathbb{Z}$. In this basis, the element ρ is given by $(0, -2, 1)$, and the simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are given by $(0, 0, 2)$ and $(1, 0, -2)$ resp.

Let u and v be formal variables representing the characters $e^{\hat{\alpha}_1}$ and $e^{\hat{\alpha}_2}$ resp. When we plug into the denominator formula, we get the Jacobi Triple product identity:

$$\sum_{m \in \mathbb{Z}} (-1)^m u^{m(m-1)/2} v^{m(m+1)/2} = \prod_{n \in \mathbb{N}} (1 - \lambda^{n+1})(1 - v\lambda^n)(1 - u\lambda^n)$$

where $\lambda = uv$.

The level one Vacuum representation for $\mathbb{T} \ltimes \tilde{L}U(n)$

Consider $\mathbb{C}^n[z, z^{-1}]$ as a real vector space with a non-degenerate symmetric bilinear form given by integrating the standard Euclidean inner product on $\mathbb{C}^n$:

$$\langle \alpha z^n, \beta z^m \rangle = \frac{1}{2\pi i} \int \langle \alpha, \beta \rangle z^{m+n-1}\, dz$$

One can now define a complex Clifford algebra $\mathcal{C}$ generated by the real vector space $\mathbb{C}^n[z, z^{-1}]$ and relations

$$f(z)g(z) + g(z)f(z) = \langle f(z), g(z) \rangle$$

Let J denote the complex structure on $\mathbb{C}^n$. Notice that the $\pm i$-eigenspaces of the complex linear extension of J yields an isotropic decomposition: $\mathbb{C}^n \otimes \mathbb{C} = \overline{W} \oplus W$. It is easy to see that W is canonically isomorphic to $\mathbb{C}^n$ as a complex vector space. This induces an isotropic decomposition of $\mathbb{C}^n[z, z^{-1}] \otimes \mathbb{C} = H_+ \oplus H_-$ with

$$H_+ = W[z] \oplus z\overline{W}[z], \quad H_- = \overline{W}[z^{-1}] \oplus z^{-1}W[z^{-1}]$$

The unique irreducible representation of $\mathcal{C}$ is given by the a Hilbert completion of the vector space $\Lambda^*(H_+)$, with H_+ acting by exterior multiplication, and H_- acting by extending the contraction operator using the derivation property. By construction, $\mathbb{T} \ltimes LU(n)$ preserves the bilinear form, and hence it acts on $\mathcal{C}$ by algebra automorphisms. Since $\Lambda^*(H_+)$ is the unique representation of $\mathcal{C}$, Schur's lemma says that we get a canonical projective action of $\mathbb{T} \ltimes LU(n)$ on $\Lambda^*(H_+)$ that intertwines the action of $\mathcal{C}$ twisted by $\mathbb{T} \ltimes LU(n)$. Lifting the map $\mathbb{T} \ltimes LU(n) \to PU(\Lambda^*(H_+))$ to the central extension yields the universal central extension $\mathbb{T} \ltimes \tilde{L}U(n)$. Notice this allows us to construct a non-trivial central extension of $\mathbb{T} \ltimes LK$ (and hence the universal one) of any compact Lie group K using an inclusion $K \subseteq U(n)$. Now let $x_1, x_2, \ldots, x_n$ denote the diagonal characters of the standard representation of $U(n)$ on $\mathbb{C}^n$. Then observe that the character of the Fermionic Fock space is given by:

$$\mathrm{Ch}(\Lambda^*(H_+)) = \prod_{m=0}^{\infty} \prod_{i=1}^{n} (1 + x_i q^m)(1 + x_i^{-1} q^{m+1})$$

The irreducible representation $L_{-\rho}$

The denominator formula is a formal identity of characters and hence we may scale each character to get the identity:

$$\sum_{w \in \hat{W}} (-1)^w e^{w(-2\rho)} = e^{-2\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{2\alpha})$$

Plugging this into the character formula gives us a formula for the character for the representation $L_{-\rho}$:

$$\frac{\sum_{w \in \hat{W}} (-1)^w e^{w(-2\rho)}}{e^{-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{\alpha})} = \frac{e^{-2\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{2\alpha})}{e^{-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 - e^{\alpha})} = e^{-\rho} \prod_{\hat{\alpha} \in \hat{\Delta}_+} (1 + e^{\alpha})$$

This character formula suggests that as a (projective) representation of $\tilde{L}K$, we have:

$$L_{-\rho} = \Lambda^*(\hat{\eta}_+)$$

One way to construct this representation is via the idea introduced in the previous example. Consider the Lie algebra of $L_{alg}K$ with its canonical inner product. On complexifying this, we get a $L_{alg}K$-invariant symmetric, non-degenerate, bilinear form on the Lie algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{h} \oplus \hat{\eta}_+ \oplus \hat{\eta}_-$. The sub spaces $\hat{\eta}_\pm$ are isotropic, dual subspaces which are orthogonal to $\mathfrak{h}$. It follows that the unique irreducible representation of the corresponding Clifford algebra is given by: $\mathbb{S} := \mathbb{S}(\mathfrak{h}) \otimes \Lambda^*(\hat{\eta}_+)$, where $\mathbb{S}(\mathfrak{h})$ is the irreducible for the (finite dimensional) Clifford sub-algebra generated by $\mathfrak{h}$.

As in the previous example, Schur's lemma shows that $L_{alg}K$ acts on $\mathbb{S}$ by projective transformations that intertwine its action on the Clifford algebra. This representation can therefore be lifted to an honest representation of $\tilde{L}_{alg}K$. On calculating the character, it is clear that $\mathbb{S}$ decomposes into the representation $L_{-\rho}$, with multiplicity equal to the dimension of $\mathbb{S}(\mathfrak{h})$.

Fusion, Conformal Blocks and the Verlinde Ring

Consider the category of positive energy representations of $\mathbb{T} \ltimes \tilde{L}K$ of level l, with the property that the space of $\hat{\eta}_-$-invariants is finite dimensional (this space is called the space of lowest weight vectors). This is another way to describe the category of representations that are finite sums of irreducibles.

Let A_l denote the Grothendieck group of this category. We have shown that this group is a finitely generated free abelian group with basis given by the weights in the Affine Alcove $l\mathrm{A}$.

$$A_l = \bigoplus_{\lambda \in l\mathrm{A}} \mathbb{Z}[L_\lambda], \quad \text{with } \lambda \text{ being a weight}$$

Fusion is a very nice geometric construction of a commutative ring structure on A_l.

Conformal Blocks:

Let $L_i, i = 1, 2, \ldots, k$ denote a collection of level l positive energy representations. Let Σ_g be a complex curve with k punctures labeled $x_i, i = 1, 2, \ldots, k$ and pick holomorphic coordinates z_i about these points. Consider the restriction homomorphism:

$$r : \mathrm{Hol}(\Sigma/\{x_i\}, G) \longrightarrow \mathrm{Hol}(\mathbb{C}^*, G)^{\times k}$$

The group on the right hand side admits a canonical central extension that extends the individual universal central extensions. However, the residue formula shows that this extension splits (canonically) when restricted along r. It follows that $\mathrm{Hol}(\Sigma/\{x_i\}, G)$ acts on $L_1 \otimes L_2 \otimes \ldots, \otimes L_k$ through the restriction. Define the space of conformal blocks to be the space of invariants: For negative energy representations, one defines it as the dual of co-invariants

$$\mathbf{C}(\Sigma, \mathbf{x}, \mathbf{L}) = (L_1 \otimes L_2 \otimes \ldots \otimes L_k)^{\mathrm{Hol}(\Sigma/\{x_i\}, G)}$$

It can be shown that this space is finite dimensional and the dimension is independent of choices. In fact, one can construct a vector bundle over the moduli space of punctured, uniformized curves of genus g, with the fiber over any curve being the space of conformal blocks. This bundle can be shown to have a projectively flat connection and it plays a very important part in conformal field theory.

Example

Consider the simplest example of $\Sigma = \mathbb{P}^1$ with one puncture x, labeled by representation L. Then $\mathrm{Hol}(\Sigma/x, G)$ is the holomorphic version of $G(\mathbb{C}[z^{-1}])$. This group contains the unipotent $\hat{U}_-$ as well as G as subgroups and hence the space of conformal blocks reduces to the space of G invariants of the space of zero-energy states V_0. Since V_0 is irreducible, this is non-zero if and only if V_0 is the trivial representation, or equivalently, if and only if L is the Vacuum representation. In the latter case, this conformal block is one dimensional. An easy generalization of this example shows that the Vacuum will be the unit in the fusion ring.

Now assume L is an irreducible positive energy representation corresponding to an irreducible V of K. Define $\overline{L}$ to be the positive energy irreducible corresponding to the conjugate K representation $\overline{V}$. This correspondence $L \mapsto \overline{L}$ will be a duality in our category. Let Σ be a curve of genus zero with three punctures labeled x_1, x_2, x_3. Pick holomorphic coordinates z_1, z_2, z_3 about these points and consider three irreducible positive energy representations: $L_1, L_2, \overline{L}_3$. Define $C_{L_1,L_2}^{L_3}$ to be the dimension of the corresponding conformal block.

Define a (bilinear) operation on A_l:

$$\mathrm{A}_l \otimes \mathrm{A}_l \longrightarrow \mathrm{A}_l, \qquad [L_1]\,[L_2] = \sum_{L_3} C_{L_1,L_2}^{L_3}\,[L_3]$$

where the sum on the right hand side runs over the (finite) set of irreducibles L_3. This bilinear operation is associative and commutative making A_l into an (Verlinde) algebra.

In fact, such operations can be defined for all genera and there is a beautiful theory of fusion rings. For example, one can use character theory over the complex numbers to derive beautiful formulas for these structure constants that involve cohomological information about the moduli of holomorphic bundles on riemann surfaces.

The map sending an irreducible K-rep with lowest weight in $l\mathrm{A}$, to the corresponding irreducible positive energy representation, extends to a surjective ring homomorphism from the representation ring of K to A_l. Its kernel is generated by exactly those irreducible K-reps V_λ with $\lambda \in C$, and satisfying $\langle \lambda - \rho_K, \alpha \rangle \in (l + \check{h})\mathbb{Z}$, for some root α of K, and where $\check{h}$ denotes the dual Coxeter number $\rho_K(h_{\alpha_0}) + 1$.

Example

The level l Verline algebra for $K = SU(2)$ is given by

$$\mathrm{A}_l = \mathbb{Z}[V]/\langle Sym^{l+1}[V] \rangle$$

where V denotes the fundamental representation of K on $\mathbb{C}^2$.