Portugal - Talk 2

Note Title 6/25/2011

Chrom level	Alg Gp	Geometric object	coh thy	
1	GL_1	mult gp	KO	J
2	GL_2	elliptic curve	TMF	Q
n	$U(1,n-1)$	ab var w/ cx mult	TAF	...

Recall:

$$\mathbb{G}_m \rightsquigarrow \text{Formal gp} \rightsquigarrow K\text{-thy}$$

$$C \rightsquigarrow \text{Formal gp} \rightsquigarrow TMF$$

Formal Gps

A 1-dim'l formal gp /R is a formal power series

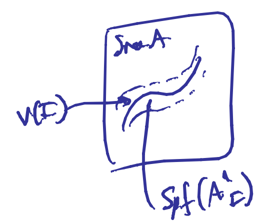
$$F(x,y) \in R[[x,y]]$$

s.t.

(1) $F(0,x) = F(x,0) = x$

(2) $F(F(x,y),z) = F(x,F(y,z))$

(3) $F(x,y) = F(y,x)$



$I \subset A$

$$\mathrm{Spec}(A)^\wedge_I = \mathrm{Spf}(A^\wedge_I) = \text{``}\varinjlim_j\text{''}\ \mathrm{Spec}(A/I^j) \qquad \text{formal scheme}\ \ \text{ind-scheme}$$

Note $\mathrm{Spf}(R[[x,y]]) \longrightarrow \mathrm{Spf}(R[[x]])$

$$F(x,y) \longleftarrow\!\shortmid\ x$$

Makes $\mathrm{Spf}(R[[x]]) = (\mathbb{A}^1_R)^\wedge_0$ into a gp object in formal schemes

$G =$

More generally, can talk about n-dim'l formal gps

Height: G has height h

if $G[p]$ is locally free of rank p^h

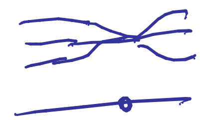
"$(\mathbb{Z}/p)^h$" ht = rank of p-torsion

$\hat{G}_m$ = mult fml gp $/R$ ht = 1 if $\operatorname{char}(R) = p$

C/R $\hat{C}$ = ht 1 or 2 if $\operatorname{char}(R) = p$

If R p-complete, over $\operatorname{Spf}(R)$

$$G \simeq \varinjlim G[p^i]$$

$G[p]$ 

$\operatorname{Spec} \mathbb{Z}_p$

Away from p: $\left.\begin{array}{l} G_m[p] \\ C[p] \end{array}\right\}$ is étale

E cx oriented Ring spectrum, G_E = fml gp ht n

E "sees" v_n-periodic phenom.

p-divisible gps

A p-divisible gp $/R$ of ht h is a sequence

$$G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$$

of finite commutative gp schemes, locally free of rank p^{ih}

s.t. $G_i = G_{i+1}[p^i]$

Think of G "=" $\varinjlim G_i$ ind-scheme

"$(\mathbb{Z}/p^\infty)^h$" $G_i = G[p^i]$

Over a field of char p, a p-divisible gp G sits in an exact sequence

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{et} \longrightarrow 0$$

G^0 = formal gp

G^{et} = étale gp scheme

$$ht(G) = ht(G^0) + ht(G^{et})$$
$$\dim(G) = \dim(G^0)$$

e.g. If A is an abelian gp scheme

$$A[p^\infty] := \varinjlim A[p^i]$$

is a p-divisible gp

Assume working over fld of char p

$\mathbb{G}_m[p^\infty]$ has ht 1, dim 1

$C[p^\infty]$ ht 2
dim = 1

$$\hat{C} \longrightarrow C[p^\infty] \longrightarrow C[p^\infty]^{et}$$

C ordinary $\Rightarrow$ ht $\hat{C} = 1$, ht $C[p^\infty]^{et} = 1$

C supersingular $\Rightarrow$ ht $\hat{C} = 2$, ht $C[p^\infty]^{et} = 0$

A = ab variety of dim d

$\mathrm{ht}\ A[p^\infty] = 2d$

$\dim A[p^\infty] = d$

<u>Formal gps $\rightsquigarrow$ Cohomology thy's</u>

<u>Thm</u> (Landweber exact functor thm)

$\mathbb{G}/R$ = formal gp

$\mathrm{Spec}(R) \xrightarrow[\mathbb{G}]{} \mathcal{M}_{FG}$ flat

$\Rightarrow \exists$ cx orientable E, $\mathbb{G}_E \cong \mathbb{G}$

<u>Cor:</u> C is an elliptic curve $/R$

$\mathrm{Spec}(R) \xrightarrow[C]{} \mathcal{M}_{ell}$ flat

E_C exists

(pf)

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r, "\text{flat}"] & \mathcal{M}_{ell} \arrow[r, "\text{flat}"] & \mathcal{M}_{FG} \\ & C \arrow[r, mapsto] & \hat{C} \end{tikzcd}$$

} apply LEFT

$\square$

<u>Problem</u> LEFT only produces a functor

$\{\text{flat FGL's}\} \longrightarrow \mathrm{Ho}(Sp)$

We need E_C to be pointset functorial in C to form sheaf $\mathcal{O}_{ell}$

Thm (Lurie)

Suppose that:

- $(A, \mathfrak{m})$ - local ring w/ perfect residue fld of char p
- $\mathfrak{X}/A$: locally noeth separated D-M stack
- $\mathbb{G} \to \mathfrak{X}$ = p-divisible gp ht h, dim 1
- $\mathfrak{X} \xrightarrow[\text{étale}]{\mathbb{G}} \mathcal{M}_{p\text{-div gps}}$

Thm

$\exists$ sheaf of E_∞-ring spectra

$\mathcal{E} / \mathfrak{X}_{ét}$

s.t. $\mathrm{Spec}(R) \xrightarrow[\text{étale}]{f} \mathfrak{X}$

Then $E = \mathcal{E}\left(\begin{smallmatrix} \mathrm{Spec}(R) \\ \downarrow \\ \mathfrak{X} \end{smallmatrix} \right)$ satisfies

- $R = \pi_0 E$
- E is weakly even periodic
  $\pi_{odd} E = 0$
  $\pi_2 E$ is invertible R-mod
- $\mathbb{G}_E \cong f^* \mathbb{G}_0$

How to check étaleness?

$$\mathfrak{X} \xrightarrow{f} \mathcal{Y} \quad \text{map of D-M stacks}$$

$\forall$ point x : $\mathrm{Spec}(k) \longrightarrow \mathfrak{X}$

$$(*) \qquad \begin{array}{ccc} \hat{\mathfrak{X}}_x & \xrightarrow{\cong} & \hat{\mathcal{Y}}_{f(x)} \\ \| & & \| \\ \mathrm{Def}_x(\mathfrak{X}) & & \mathrm{Def}_{f(x)}(\mathcal{Y}) \end{array}$$

$\Big(\forall$ complete local R

$k \to R/m$

$$\mathrm{Def}_x(\mathfrak{X})(R) = \left\{ \begin{array}{ccc} \mathrm{Spec}(k) & \to & \mathfrak{X} \\ \downarrow & \nearrow & \\ \mathrm{Spf}(R) & & \end{array} \right\} \Big)$$

$\underline{\text{Eg.}}$

$\mathfrak{X}$ = moduli stack of multiplicative gps $/\mathbb{Z}_{(p)}$

$\| $

$\mathcal{M}_{\mathbb{G}_m} \qquad \{\mathbb{G}_m\} \circlearrowleft_{z \mapsto z^{-1}}$

$\mathbb{G} = \mathbb{G}_m[p^\infty]$

$\underline{\text{Fact:}}$ $\mathbb{G}_m[p^\infty]$, $\mathbb{G}_m$ have unique deformations

condition $(*)$ trivially satisfied

$$\mathrm{Spec}(\mathbb{Z}_{(p)}) \xrightarrow[\mathbb{G}_m]{C_2} \mathcal{M}_{\mathbb{G}_m}$$

$$\mathcal{E}(-) = K_{(p)}$$

$$\Gamma\mathcal{E} = K_{(p)}^{hC_2} = KO_{(p)}$$

Eg. $\mathcal{X} = (\mathcal{M}_{ell})_{(p)}$ = moduli stack of elliptic curves (localized to $\mathbb{Z}_{(p)}$)

$$\begin{array}{c} C_{univ} \\ \downarrow \\ (\mathcal{M}_{ell})_{(p)} \end{array} \qquad \mathbb{G} = C_{univ}[p^\infty]$$

(*) Thm (Serre-Tate)

$$\{\text{Deformations of } C\} \xrightarrow{\simeq} \{\text{Deformations of } C[p^\infty]\}$$

Lurie's thm:

$$\Rightarrow \quad \mathcal{E} = \mathcal{O}_{ell}$$

$$\mathcal{E}\left(\begin{array}{c} \mathrm{Spec}(R) \\ \downarrow^{C} \\ \mathcal{M}_{ell} \end{array}\right) = E_C$$

$$\Gamma\mathcal{E} = TMF$$

Eg. $\mathbb{G}_0$ = ht n formal gp / $\overline{\mathbb{F}}_{p^n}$

Lubin-Tate showed $\mathrm{Def}_{\mathbb{G}_0} = \mathrm{Spf}(R) \;\Big|\; \mathbb{G}/R$ univ. def.

$$\mathcal{X} = (\mathcal{M}_{p\text{-div}})^\wedge_{\mathbb{G}_0} = \mathrm{Spf}(R) /\!\!/ \mathrm{Aut}(\mathbb{G}_0)$$

(*) tautologically satisfied

$$\mathcal{E}\left(\mathrm{Spf}(R) \xrightarrow[\substack{\uparrow \\ \text{ind-etale}}]{\mathbb{G}} (\mathcal{M}_{p\text{-div}})^{\wedge}_{\mathbb{G}_0}\right) = E_n$$

Morava E-thy

formal gp $\mathbb{G}$

$$\mathbb{S}_n = \mathrm{Aut}(\mathbb{G}_0) \circlearrowright E_n$$

"Lurie's thm is a kind of generalization of Hopkins–Miller thm"

$\underline{\underline{EO_n}}$

In fact, the first "versions" of TMF appeared as consequences of E_n-thy:

$EO_n := E_n^{hG}$ $\quad G \leq \mathbb{S}_n$ maximal finite

(N.B. Non-unique for $n \gg 0$)

e.g. $EO_1 = KO^{\wedge}_p$ $\quad p = 2$

$EO_2 = TMF_{K(2)}$ $\quad p = 2, 3$

EO_n gives a fun "$K(n)$-local" higher analog of TMF

Moduli of higher dim'l ab varieties?

Problem: don't give 1-dim'l formal gps

Solution: add data to split off 1-dim'l summand.
(well studied in # thy)

Shimura Varieties

Fix the following initial data;

$$\begin{array}{c} F \\ \big|\,{\scriptstyle 2} \\ \mathbb{Q} \end{array} \quad \text{quadratic imaginary extension,} \qquad \begin{array}{c} \mathfrak{p}\bar{\mathfrak{p}} \\ \big| \\ p \end{array}$$

$\mathcal{O}_F$ = ring of integers

V = F-v.s. dim n

$\langle -,- \rangle : V \otimes_{\mathbb{Q}} V \longrightarrow \mathbb{Q}$ alternating herm form

$$\langle x,y \rangle = -\langle y,x \rangle$$

$$\langle ax,y \rangle = \langle x, \bar{a} y \rangle$$

$$\langle x,y \rangle = \mathrm{Tr}_{F/\mathbb{Q}}(x \beta y^*) \qquad \text{Signature } (1,n-1)$$

$$\beta^* = -\beta \qquad \begin{bmatrix} e_1 i & & & \\ & -e_2 i & & \\ & & \ddots & \\ & & & -e_n i \end{bmatrix}$$

L = $\mathcal{O}_F$ - lattice in V

$$\langle L,L \rangle \subseteq \mathbb{Z}$$

$$U(\mathbb{C}) = U(1,n-1)$$

U = unitary gp $/\mathbb{Q}$

$$U(R) = \{ g \in \mathrm{Aut}_F(V) \otimes R \mid \langle gx, gy \rangle = \langle x,y \rangle \}$$

GU = unitary similitude gp $/\mathbb{Q}$

$$GU(R) = \{ g \in \mathrm{Aut}_F(V) \otimes R \mid \langle gx, gy \rangle = \nu(g) \langle x,y \rangle, \ \nu(g) \in R^\times \}$$

Shimura variety

$\mathrm{Sh}_u / \mathrm{Spf}(\mathbb{Z}_p)$ is a stack

$\mathrm{Sh}_u(S) \in \mathrm{Gpd}.$

Objects: (A, i, λ)

A = ab scheme$/S$ dim n	Subject to two conditions
$\lambda: A \to A^\vee$ polarization, compatible	$(*)$
$i: \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$	$(**)$

Polarization?

$$\lambda: A \xrightarrow{\text{isog}} A^\vee \quad \longleftrightarrow \quad \begin{matrix} W \to W^* \\ W \otimes W \to k \end{matrix} \quad \text{bilinear form}$$

polarization "symmetric pos def"

A/k $\quad T_\ell A = \varprojlim_i A(\bar{k})[\ell^i] \cong \mathbb{Z}_\ell^{2d}$ $\quad \dim A = d$

$\lambda: T_\ell A \longrightarrow T_\ell A^*$

$\langle -, - \rangle_\lambda : T_\ell A \otimes T_\ell A \longrightarrow \mathbb{Z}_\ell$ $\quad$ Weil-pairing

λ also induces a Rosati involution:

$$\dagger \circlearrowright \operatorname{End}(A) \otimes \mathbb{Q}$$

$$\begin{tikzcd} A \arrow[r, "\lambda"] & A^\vee \\ A \arrow[u, dashed, "\varphi^\dagger"] \arrow[r, "\lambda"'] & A^\vee \arrow[u, "\varphi^\vee"'] \end{tikzcd}$$

$\varphi \in \operatorname{End} A$
$\rightsquigarrow$
$\varphi^\dagger \in \operatorname{End} A$

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compatible means $i(\bar{a}) = i(a)^\dagger$

$(*)_p$ $\quad i : \mathcal{O}_F \longrightarrow \operatorname{End}(A) \longrightarrow \operatorname{End}(A[p^\infty])$

$$\begin{tikzcd} \mathcal{O}_F \arrow[r] \arrow[dr] & \operatorname{End}(A) \arrow[r] & \operatorname{End}(A[p^\infty]) \\ & \mathcal{O}_{F,p} \arrow[ur, dashed] & \end{tikzcd}$$

$$\mathcal{O}_{F,p} \cong \mathcal{O}_{F,u} \times \mathcal{O}_{F,\bar{u}}$$

$$\Longrightarrow \quad A[p^\infty] \cong A[u^\infty] \oplus A[\bar{u}^\infty]$$

$$\dim A[u^\infty] = 1$$
$$(\dim A[\bar{u}^\infty] = n-1)$$

$(**)_\ell$ $\quad \forall\ \ell \neq p,\ \exists$ similitudes of $\mathbb{Z}_\ell$-modules

$$\eta_\ell : (L_\ell, \langle -,- \rangle) \xrightarrow{\sim} (T_\ell V, \langle -,- \rangle_\lambda)$$

Consider:

$$\mathrm{Sh}_{\mathcal{U}} \longrightarrow \mathcal{M}_{p\text{-div}}$$

$$(A, i, \lambda) \longmapsto A[u^\infty]$$

Claim: this map is étale

Pf Need to show

$$\{\text{Deformations of } (A,i,\lambda)\} \xrightarrow{\simeq} \{\text{Deformations of } A[u^\infty]\}$$

Serre-Tate thy says

$$\text{Deformations of } A \longleftrightarrow \text{Deformations of } A[p^\infty]$$

$$\text{Deformations of } (A,i) \longleftrightarrow \begin{array}{l}\text{Deformations of } A[p^\infty] \text{ which preserve decomp} \\ A[p^\infty] \cong A[u^\infty] \oplus A[\bar{u}^\infty] \\ \quad \wr\wr \\ \text{Def of } A[u^\infty] \\ \quad \times \text{ Def of } A[\bar{u}^\infty]\end{array}$$

$$(A, i, \lambda) \qquad \lambda: A[\bar{u}^\infty]^\vee \xrightarrow{\simeq} A[u^\infty]$$

$$\left(\begin{array}{c}\text{Def's of} \\ A, i, \lambda\end{array}\right) \longleftrightarrow \text{Defs } (A[u^\infty])$$

$\square$

Applying Lurie's thm to $(Sh_U, A_{univ}[u^\infty])$

$\exists$ $\mathcal{E}_U$ sheaf of E_∞-algs on Sh_U

$$\mathcal{E}_U\left(\begin{array}{c} Spf(R) \\ \downarrow (A,i,\lambda) \\ Sh_U \end{array}\right) = E$$

$$\pi_0 E = R, \quad \mathbb{G}_E = A[u^\infty]_0$$

$$TAF := \Gamma \mathcal{E}_U$$

Relation to automorphic forms:

$$\begin{array}{c} \mathrm{Lie}\, A \text{ is } p\text{-complete} \\ \cong \\ (\mathrm{Lie}\, A)_u \oplus (\mathrm{Lie}\, A)_{\bar{u}} \\ \uparrow \\ \text{locally free of rank 1} \end{array}$$

$$\begin{tikzcd} \omega \arrow[d] & \omega_{(A,i,\lambda)} = (\mathrm{Lie}^* A)_u \arrow[d] \\ Sh_U & \ni (A,i,\lambda) \end{tikzcd}$$

Descent spectral sequence

$$H^s(Sh_U; \omega^{\otimes t}) \Rightarrow \pi_{2t-s}\, TAF$$

$(s=0 \Rightarrow$ "integral Automorphic forms")

$\mathbb{C}$-points

$$\begin{tikzcd} Sh_K & \widetilde{Sh}_K \arrow[l] \arrow[r] & Sh_K^{\mathbb{C}} \\ \mathbb{Z}_p & \mathbb{Z}_{(p)} \arrow[l] \arrow[r] & \mathbb{C} \end{tikzcd}$$

$$\widetilde{Sh}_K(\mathbb{C}) \simeq (A, i, \lambda)$$

$A =$ n-dim'l ab variety $/\mathbb{C}$

$i : O_F \hookrightarrow \operatorname{End}(A)$

$\lambda : A \longrightarrow A^\vee$ polarization

$\operatorname{Lie} A = \mathbb{C}$-v.s.

$\circlearrowleft$

O_F $\qquad\qquad O_F \otimes \mathbb{R} = \mathbb{C}$

Two different $\mathbb{C}$ structures!

$$\operatorname{Lie} A = \operatorname{Lie} A^+ \oplus \operatorname{Lie} A^-$$

Condition

$(*)$ $\operatorname{Dim}_{\mathbb{C}} \operatorname{Lie} A^+ = 1$

$(**)$ $\forall\, \ell,\ (T_\ell(A), \langle -,- \rangle_\lambda) \sim (L_\ell, \langle -,- \rangle)$

Model

$V_\infty := V \otimes \mathbb{R}$

$\mathcal{H} =$ space of compatible $\mathbb{C}$ structures on V_∞ $= \left\{ J \underset{F}{\circlearrowright} V_\infty \;\middle|\; \begin{array}{l} J^2 = -1 \\ \langle -, J - \rangle \text{ symmetric + positive} \end{array} \right\}$

given $J \in \mathcal{H}$

get $A_J = V_\infty / L$,

$i : \mathcal{O}_F \to \mathrm{End}(A_J)$

$\langle -, - \rangle$ = Riem form on V_∞ $\longleftrightarrow$ polarization on A_J

$\langle -, J- \rangle + i\langle -, - \rangle : V_\infty \to V_\infty^*$

$U(1, n-1) = U(\mathbb{R}) \circlearrowright \mathcal{H}$

stabilizer K_∞ maximal cpt

$\cong$

$U(1) \times U(n-1)$

$\mathcal{H} = U(\mathbb{R}) / K_\infty$

$\widetilde{Sh}_u(\mathbb{C}) \simeq \coprod_{L'} \Gamma_{L'} \backslash \mathcal{H}$

lattice class grp

$\Gamma \subset GU(\mathbb{Q})^+$

preserve lattice

<u>Given</u> V_ρ = f.d. $\mathbb{C}$-rep of K_∞

get

$$\begin{array}{ccc} \mathcal{V}_\rho & & \coprod_{L'} \Gamma_{L'} \backslash U(\mathbb{R}) \times_{K_\infty} V_\rho \\ \downarrow & & \downarrow \\ \widetilde{Sh}_u(\mathbb{C}) & & \coprod_{L'} \Gamma_{L'} \backslash U(\mathbb{R}) / K_\infty \end{array}$$

Hol automorphic forms of wt ρ_κ

$\updownarrow$

hol sections of V_ρ

$w \longleftrightarrow \rho : U(1) \times U(\infty) \longrightarrow U(1) \circlearrowright \mathbb{C}$

$\}$ wt -1

(scalar wt)