Deformed special geometry: the Hesse potential and the holomorphic anomaly equation

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This talk is about:

deformations of special geometry

and the relation with topological string theory.

Central role: the Hesse potential of real special geometry

Introduction

- $N = 2$ Wilsonian action: in the presence of Weyl^2 terms, encoded in

$$F(Y, \Upsilon) = \sum_{g=0}^{\infty} \Upsilon^g F^{(g)}(Y)$$

Y^I complex scalar fields , $\Upsilon \sim$ Weyl^2.

Duality (symplectic) transformations act on full vector

$$(Y^I, F_I(Y, \Upsilon)) \quad , \quad F_I = \frac{\partial F(Y, \Upsilon)}{\partial Y^I}$$

- Topological string theory: perturbative free energy

$$F(\mathcal{Y}, \lambda) = \sum_{g=0}^{\infty} \lambda^{g-1} F^{(g)}(\mathcal{Y})$$

Duality (symplectic) transformations act on $(\mathcal{Y}^I, F_I^{(0)}(\mathcal{Y}))$.
$F^{(g)}(\mathcal{Y})$ $(g \geq 1)$ transform as functions.

Introduction

- Both expansions defined in terms of different variables.
- In addition: both LEEA and topological string receive non-holomorphic corrections.

Proposal:

LEEA and TST related through the Hesse potential of real special geometry, $H(\phi, \chi)$.

- (ϕ, χ): same type of variables as those of TST.
- TST coincides with part of $H(\phi, \chi)$.
- $H(\phi, \chi)$ is Legendre transform of $\mathrm{Im}F(Y, \Upsilon)$ (LEEA).
- Includes non-holomorphic corrections: consistent deformation of special geometry.

Deformed special geometry

EA-side: consistent non-holomorphic extension

$$F(Y,\Upsilon) \longrightarrow F = F(Y,\Upsilon) + 2i\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$$

where Ω is real.

Duality (symplectic) transformations act on full vector

$$(Y^I, F_I) \quad , \quad F_I = \frac{\partial F}{\partial Y^I}$$

- scalar manifold : intrinsic torsion
  (with `Alvaro Osorio, arXiv: arXiv:1212.4364`)
- Specific deformation yields holomorphic anomaly equation of TST.

Deformation of special geometry

Classical mechanics system: n dof $\quad i = 1, \ldots n$

coordinates ϕ^i, velocities $\dot{\phi}^i$, Lagrangian $L(\phi, \dot{\phi})$

Hamiltonian $H(\phi, \pi) = \dot{\phi}^i \, \pi_i - L(\phi, \dot{\phi})$. Patch of phase space (ϕ^i, π_i).

Complex coordinates $z^i = \frac{1}{2}\left(\phi^i + i\,\dot{\phi}^i\right)$.

Theorem: $\exists$ function

$$F(z, \bar{z}) = F^{(0)}(z) + 2i\,\Omega(z, \bar{z}) \quad , \quad \Omega \ \text{real}$$

$$\begin{pmatrix} \phi^i \\ \pi_i \end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix} z^i \\ F_i(z, \bar{z}) \end{pmatrix} \quad , \quad F_i = \frac{\partial F(z, \bar{z})}{\partial z^i}$$

Equivalence relation: $F(z, \bar{z}) \rightarrow F(z, \bar{z}) + \bar{g}(\bar{z})$

$$F^{(0)}(z) \rightarrow F^{(0)}(z) + g(z) \quad , \quad \Omega \rightarrow \Omega - \mathrm{Im}\, g(z)$$

Deformation of special geometry

(z^i, F_i) complexification of phase space coordinates (ϕ^i, π_i):

canonical transformations = $\mathrm{Sp}(2n, \mathbb{R})$ transformations

$$\begin{pmatrix} z^i \\ F_i(z,\bar z) \end{pmatrix} \to \begin{pmatrix} \tilde z^i \\ \tilde F_i(\tilde z,\bar{\tilde z}) \end{pmatrix} = \begin{pmatrix} U^i_j & Z^{ij} \\ W_{ij} & V_i{}^j \end{pmatrix} \begin{pmatrix} z^j \\ F_j(z,\bar z) \end{pmatrix}$$

Transformation is integrable: $F(z,\bar z) \to \tilde F(\tilde z, \bar{\tilde z})$.

$$\begin{aligned} L &= 4\,[\mathrm{Im}\, F - \Omega] \\ H &= -i\left(z^i\,\bar F_{\bar\imath} - \bar z^{\bar\imath}\,F_i\right) - 2\left(2\Omega - z^i\Omega_i - \bar z^{\bar\imath}\,\Omega_{\bar\imath}\right) \\ &\quad -4\mathrm{Im}\left[F^{(0)} - \tfrac{1}{2} z^i F^{(0)}_i\right] \end{aligned}$$

H is a symplectic function: $\tilde H(\tilde\phi, \tilde\pi) = H(\phi, \pi)$.

When $\Omega(z,\bar z) = w(z) + \bar w(\bar z)$: $F(z) = F^{(0)}(z) + 2i\,w(z)$

Evaluating the Hesse potential

In $N = 2$ supergravity, H is the Hesse potential.

Hesse potential $H(\phi, \pi)$ is the Legendre transform of $\mathrm{Im}\, F - \Omega$, where

$$F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$$

New variables:

$$\begin{pmatrix} \phi^I \\ \pi_I \end{pmatrix} = 2\,\mathrm{Re} \begin{pmatrix} Y^I \\ F_I(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \end{pmatrix} = 2\,\mathrm{Re} \begin{pmatrix} \mathcal{Y}^I \\ F_I^{(0)}(\mathcal{Y}) \end{pmatrix}$$

where

$$\begin{aligned} \mathcal{Y}^I &= Y^I + \Delta Y^I(\Omega) \quad , \quad \Omega \neq 0 \\ \mathcal{Y}^I &= Y^I \quad , \quad \Omega = 0 \,. \end{aligned}$$

Evaluate H in terms of $\mathcal{Y}^I \Rightarrow$ power series expansion in $\Omega(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon, \bar{\Upsilon})$.

Evaluating the Hesse potential

The Hesse potential transforms as a function under symplectic transformations: $\tilde{H}(\tilde{\phi},\tilde{\pi}) = H(\phi,\pi)$.

- H as series of symplectic functions, $H = \sum_{k=0}^{\infty} H^{(k)}(\mathcal{Y},\bar{\mathcal{Y}},\Upsilon,\bar{\Upsilon})$
- $H^{(0)} = -i\left[\bar{\mathcal{Y}}^I F_I^{(0)}(\mathcal{Y}) - \text{c.c.}\right]$
- $H^{(1)}$ is the only one that contains $\Omega(\mathcal{Y},\bar{\mathcal{Y}},\Upsilon,\bar{\Upsilon})$ (the other $H^{(k)}$ contain derivatives thereof)

$$H^{(1)} = 4\Omega - 4N^{IJ}\left(\Omega_I\Omega_J + \Omega_{\bar{I}}\Omega_{\bar{J}}\right) + \mathcal{O}(\Omega^3)$$
$$N_{IJ} = -i\left(F_{IJ}^{(0)} - \bar{F}_{\bar{I}\bar{J}}^{(0)}\right)$$

Diagrammatic expansion in terms of tree graphs. Propagator N_{IJ}.

- $N^{IJ} \to \ldots(N - \mathrm{i}\mathcal{Z})^{IJ}$

$$\tilde{\Omega}(\tilde{\mathcal{Y}},\bar{\tilde{\mathcal{Y}}}) = \Omega - \mathrm{i}\left(\mathcal{Z}^{IJ}\,\Omega_I\Omega_J - \bar{\mathcal{Z}}^{\bar{I}\bar{J}}\,\Omega_{\bar{I}}\Omega_{\bar{J}}\right) + \mathcal{O}(\Omega^3)$$

So far, general. Now pick

- $\Upsilon \in \mathbb{R}$,
- $\Omega(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon) = \Upsilon \left(w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha \ln \det[N_{IJ}] \right) + \mathcal{O}(\Upsilon^2)$

  $\longrightarrow$ function at $\mathcal{O}(\Upsilon)$. At higher order:

  diagrammatic expansion in terms of connected loop graphs.
- Expanding in powers of Υ:

$$H^{(1)} = \sum_{g=1}^{\infty} \Upsilon^g \left[F^{(g)}(\mathcal{Y}, \bar{\mathcal{Y}}) + \mathrm{h.c.} \right]$$

  $F^{(g)}$ are symplectic functions that satisfy the holomorphic anomaly equation of topological string theory ($g \geq 2$)

$$\partial_{\bar{I}} F^{(g)} = i \, \bar{F}^{(0)}_{\bar{I}\bar{P}\bar{Q}} N^{PJ} N^{QK} \left(2\alpha \, D_J \partial_K F^{(g-1)} + \sum_{r=1}^{g-1} \partial_J F^{(r)} \partial_K F^{(g-r)} \right)$$

Conclusions and Outlook

- Consistent deformation of special geometry $\longleftrightarrow$ perturbative TST (holomorphic anomaly equation)
- perturbative TST captured by $H^{(0)}$ and $H^{(1)}$ (part of Hesse potential)

  (higher $H^{(k)}$ are derived functions)
- Power series in Υ. Not convergent. Need to incorporate $\mathrm{e}^{-1/\Upsilon}$ effects $\longleftrightarrow$ non-perturbative completion of TST.
- BH partition function

$$Z(\phi, \chi) = \sum_{q,p} d(q,p)\, \mathrm{e}^{\pi[q\phi - p\chi]} \sim \mathrm{e}^{H(\phi,\chi)} = \mathrm{e}^{H^{0)}+H^{(1)}+\ldots}$$

Thanks!