Exercise – The Euler-Heisenberg Lagrangian for QED

Consider the effective action that arises when integrating out a fermion of mass m. When the electric-magnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is constant, the effective Lagrangian is called the Euler-Heisenberg Lagrangian.

The Euler-Heisenberg Lagrangian is determined in terms of the Schwinger integral

$$\int_\epsilon^\infty \frac{ds}{s}\, e^{-s\,m^2}\, \mathrm{Tr}\, e^{-s\,\hat{H}} \ , \tag{1}$$

where

$$\hat{H} = (\hat{p} - eA)^2 + \tfrac{e}{2}\, \sigma^{\mu\nu}\, F_{\mu\nu} \ .$$

Show that the Schwinger integral (1) is proportional to

$$F_{\mu\nu}\tilde{F}^{\mu\nu} \int_\epsilon^\infty \frac{ds}{s}\, e^{-s\,m^2}\, \frac{\mathrm{Re}\ \cosh(se\,X)}{\mathrm{Im}\cosh(se\,X)}$$

for a suitably defined quantity X. Determine X.