
Exercise 1 – Renormalization in scalar ϕ^3 theory in $\mathbb{R}^{1,3}$

Consider the Lagrangian

$$L = -\tfrac{1}{2}\,\phi\Box\phi + \frac{g}{3!}\,\phi^3 \ .$$

Compute the one-loop correction

$$\mathcal{M}_{1-loop} = (ig)^2 \frac{i}{p^2}\,\tilde{\mathcal{M}}\,\frac{i}{p^2}$$

to the tree-level ϕ-propagator $i\mathcal{M}_{tree} = (ig)^2\, i/p^2$, where

$$i\tilde{\mathcal{M}} \equiv \tfrac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{(k-p)^2+i\epsilon}\,\frac{i}{k^2+i\epsilon} \quad , \quad \epsilon > 0 \ .$$

Proceed as follows.

a) Let $A, B \in \mathbb{C}\backslash\{0\}$. Using

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{[A+(B-A)x]^2} \ ,$$

show that

$$i\tilde{\mathcal{M}} = \tfrac{1}{2}g^2 \int \frac{d^4k}{(2\pi)^4}\int_0^1 dx\,\frac{1}{(k^2-\Delta+i\varepsilon)^2} \ ,$$

where $\Delta = -p^2x(1-x)$.

b) Take $p^2 < 0$, so that $\Delta > 0$ for $0 < x < 1$. On the next assignment you will be asked to prove the following relation,

$$\lim_{\varepsilon\to 0}\left(\int \frac{d^4k}{(2\pi)^4}\frac{2}{(k^2-\Delta+i\varepsilon)^3}\right) = -\frac{i}{16\pi^2\Delta} \ .$$

Integrating this with respect to Δ gives

$$-\frac{i}{16\pi^2}\ln\frac{\Delta}{\Lambda^2} \equiv I(\Delta) \ ,$$

where Λ is an integration constant that plays the role of an UV-cutoff. Using this result, we regularize the integral

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^2}$$

by introducing the cutoff Λ, to obtain

$$\int_{regularized} \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^2} = I(\Delta) \ .$$

c) Now consider the regularized one-loop amplitude $\tilde{\mathcal{M}}$ with $p^2 < 0$. Show that

$$\tilde{\mathcal{M}}^{regularized} = \frac{g^2}{32\pi^2}\left(2 - \ln\frac{-p^2}{\Lambda^2}\right) \ .$$

Bring this into the final form

$$\tilde{\mathcal{M}}^{regularized} = -\frac{g^2}{32\pi^2}\,\ln\frac{-p^2}{\Lambda^2}$$

by performing an appropriate rescaling of Λ.

d) Setting $Q^2 = -p^2 > 0$, we thus obtain for the 1-loop corrected propagator,

$$\mathcal{M}(Q) = \mathcal{M}_{tree} + \mathcal{M}_{1-loop} = \frac{g^2}{Q^2}\left(1 - \frac{1}{32\pi^2}\frac{g^2}{Q^2}\ln\frac{Q^2}{\Lambda^2}\right) .$$

Now recall that in scalar ϕ^3 theory, g has dimension of mass. Therefore, we introduce the dimensionless coupling $\tilde{g}^2 = g^2/Q^2$, and obtain

$$\mathcal{M}(Q) = \tilde{g}^2\left(1 - \frac{1}{32\pi^2}\tilde{g}^2\ln\frac{Q^2}{\Lambda^2}\right) .$$

Next, pick a scale $Q_0 < \Lambda$ and define the renormalized coupling $\tilde{g}_R$ at this scale to be $\tilde{g}_R^2 = \mathcal{M}(Q_0)$. Show that, when expressed in terms of $\tilde{g}_R$, $\mathcal{M}(Q)$ takes the form

$$\mathcal{M}(Q) = \tilde{g}_R^2\left(1 + \frac{1}{32\pi^2}\tilde{g}_R^2\ln\frac{Q_0^2}{Q^2}\right) + \mathcal{O}(\tilde{g}_R^6) .$$

Viewing $\mathcal{M}(Q)$ as an effective dimensionless coupling $\tilde{g}_{eff}^2(Q)$, we establish that $\tilde{g}_{eff}$ runs with momentum. Compute the associated beta function

$$Q_0\frac{d\tilde{g}_{eff}(Q_0)}{dQ_0} = \beta(\tilde{g}_{eff}(Q_0)) .$$