Exercise 1 – A variant of the Trotter formula

Let $M_n(\mathbb{C})$ denote the space $n \times n$ matrices with complex entries. Let $A \in M_n(\mathbb{C})$ satisfy $||A - \mathbb{I}|| < 1/2$, where $\mathbb{I}$ denotes the identity matrix, and where $|| \cdot ||$ denotes the Hilbert-Schmidt norm. Define

$$\ln A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - \mathbb{I})^m}{m} ,$$

which is continuous on the set of matrices $A \in M_n(\mathbb{C})$ with $||A - \mathbb{I}|| < 1/2$, and satisfies $\exp \ln A = A$.

a) Show that there exists a constant c such that

$$|| \ln A - (A - \mathbb{I})|| \leq c \, ||A - \mathbb{I}||^2 .$$

Thus, $\ln A = A - \mathbb{I} + O(||A - \mathbb{I}||^2)$.
b) Use this result to show that as $N \to \infty$,

$$\ln \left(e^{A/N} \, e^{B/N} \right) = \frac{A}{N} + \frac{B}{N} + O(N^{-2}) .$$

c) Conclude

$$\lim_{N \to \infty} \left(e^{A/N} \, e^{B/N} \right)^N = e^{A+B} .$$

Exercise 2 – Differential operators and determinants

Consider the following second-order differential operators on the interval $[0, T]$ with Dirichlet boundary conditions: $A(\omega) = -D^2 - \omega^2$, $B(\omega) = -D^2 + i\omega \, D$. Show that $\det B(2\omega) = \det A(\omega)$.

Exercise 3 – Propagator of quantum particle in constant uniform magnetic field $\vec{B} = (0, 0, B)$

Consider the Lagrangian for a particle in a constant uniform magnetic field $\vec{B} = (0, 0, B)$,

$$L = \frac{m}{2} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \frac{eB}{2} \left(x \, \dot{y} - y \, \dot{x} \right) .$$

Using the result of the previous exercise, show that the propagator of the quantum particle is given by

$$\begin{aligned} K(\vec{q}\,', T; \vec{q}, 0) &= \left(\frac{m}{2\pi i \hbar T} \right)^{3/2} \frac{\omega T}{\sin(\omega T)} \exp \left[\frac{im}{2\hbar} \left(\frac{(z' - z)^2}{T} \right. \right. \\ & \left. \left. + \omega \cot(\omega T) \left((x' - x)^2 + (y' - y)^2 \right) + 2\omega (x \, y' - y \, x') \right) \right] , \end{aligned}$$

with a specific value for ω.