Exercise – Renormalization in one-dimensional quantum mechanics

We seek a low-energy theorem for the scattering from a short-range potential in one spatial dimension. We take this potential to have a width of order a. We represent this potential in terms of distributions given by delta functions.

Consider the time-independent Schrödinger equation in one spatial dimension,

$$-\frac{1}{2}\,\Psi''(x) + V(x)\,\Psi(x) = E\,\Psi(x)\ , \tag{1}$$

where $E > 0$ and where $V(x)$ denotes a short-range potential given by the distribution

$$V(x) = c\,\frac{\delta(x+a) - \delta(x-a)}{2\,a}\ . \tag{2}$$

The paramenter a is a short-distance cutoff. It defines a potential with width of order a. The parameter c is dimensionless.

We seek a solution of (1) of the form

$$\Psi(x) = \begin{cases} A\,e^{ipx} + B\,e^{-ipx}\ , & x < -a \\ \tilde{A}\,e^{ipx} + \tilde{B}\,e^{-ipx}\ , & -a < x < a \\ C\,e^{ipx}\ , & x > a \end{cases}$$

where $p = \sqrt{2E} > 0$.

a) Imposing continuity and the appropriate jump conditions at $x = -a$ and $x = a$, show that the transmission coefficient $T \equiv C/A$ is given by

$$\frac{1}{T} = 1 + \frac{c^2}{4\,a^2\,p^2}\left(1 - e^{4iap}\right)\ . \tag{3}$$

b) Show that at low-energies, i.e. $p\,a \ll 1$, the transmission probability is given in terms of a renormalized parameter c_R as

$$|T|^2 = \frac{p^2}{c_R^2}\ . \tag{4}$$

Determine c_R and verify that it is dimensionful.
Thus, the dimensionless parameter c has been traded for a dimensionful parameter c_R. Verify that the result (4) is independent of the short-distance cutoff a in that scalings of a can be compensated by scalings of c.