Exercise 1 – Superficial degree of divergence of a 1PI graph

The superficial degree of divergence D of a 1PI graph is defined by the power of the overall momentum dependence of the diagram.

a) Consider a ϕ^4-theory in d dimensions. Each loop integral contributes k^d, while an internal line contributes k^{-2}. The superficial degree of divergence of a 1PI graph is therefore

$$D = d\,L - 2\,I \ ,$$

where L denotes the number of loop integrals and I denotes the number of internal lines.
Let E and V denote the number of external lines and of vertices of the graph, respectively,
Show that

$$L = I - (V-1) \quad , \quad 4V = E + 2I \quad , \quad D = (d-4)L + 4 - E \ .$$

Thus, in four dimensions, $D = 4 - E$.

b) The superficial degree of divergence of a 1PI graph in spinor QED is

$$D = 4 - N_\gamma - \frac{3}{2}\,N_e \ ,$$

where $N_\gamma (N_e)$ denotes the number of external photon (fermion) lines. Which are the superficially divergent 1PI graphs at 1-loop with $D \geq 0$?

Exercise 2 – RG flow fixed points

Consider a scalar field theory in $d = 4 - \epsilon$ dimensions (with $0 \leq \epsilon \leq 1$) with couplings $g_2\phi^2$ and $g_4\phi^4$, and beta functions given by

$$\Lambda\frac{dg_2}{d\Lambda} = -2g_2 - \frac{1}{16\pi^2}\,\frac{g_4}{1+g_2} \quad , \quad \Lambda\frac{dg_4}{d\Lambda} = -\epsilon g_4 + \frac{3}{16\pi^2}\,\frac{g_4^2}{(1+g_2)^2} \ .$$

a) Determine the fixed points of these equations.

b) Consider the linearized beta functions that one obtains by expanding around these fixed points, $g_2 = g_2^* + \delta g_2, g_4 = g_4^* + \delta g_4$,

$$\Lambda\frac{d\delta g_i}{d\Lambda} = M_{ij}\delta g_j \ .$$

Determine the eigenvectors and the eigenvalues of the matrix M_{ij}.

c) Discuss the nature of the fixed points: how many relevant/irrelevant flow directions are there when flowing towards the IR?