Exercise – The Casimir force

Use heat-kernel regularization to compute the Casimir effect in one spatial dimension ($\hbar = c = 1$).

a) Compute

$$E(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n \, e^{-n\pi/(d\Lambda)} \ ,$$

with the UV-regulator Λ satisfying $d\Lambda \gg 1$.

b) Next, compute the total energy $E_{\text{total}} = E(d) + E(L-d)$, and exhibit its dependence on the UV-regulator Λ.

c) Compute the Casimir force in the limit $L \to \infty$ (removal of the IR-regulator),

$$F(d) = -\frac{\pi}{24d^2} \ .$$

Exercise – Renormalizability of theories

Consider first the action for a scalar bosonic field ϕ in $\mathbb{R}^n$,

$$S(\phi) = \int \left(\frac{1}{2} (\nabla\phi)^2 + \frac{m^2}{2} \phi^2 + \sum_{k\geq 3} g_k \, \phi^k \right) d^n x \ .$$

We denote the mass dimension of the coupling g_k by $[g_k]$. An interaction term ϕ^k is called subcritical if $[g_k] > 0$, and critical if $[g_k] = 0$. A field theory is called superrenormalizable if all interaction terms in $S(\phi)$ are subcritical, and is called renormalizable if all interaction terms are critical or subcritical. Otherwise, the theory is called unrenormalizable.

a) Which interaction terms are subcritical for $n = 2, 3, 4, 5, 6$?

b) Are the theories $(n = 3, k = 6)$, $(n = 4, k > 4)$, $(n = 5, k \geq 4)$, $(n = 6, k = 3)$ renormalizable?

Now consider a theory with a bosonic scalar field ϕ and a fermionic field ψ in $\mathbb{R}^n$,

$$S(\phi, \psi) = \int \left(\frac{1}{2} (\nabla\phi)^2 + (\psi, D\psi) + g \, \phi \, [\psi, \psi] \right) d^n x \ ,$$

where $(\psi, D\psi)$ denotes the kinetic term for ψ, and $\phi \, [\psi, \psi]$ is called Yukawa interaction.

c) Compute the mass dimension of the coupling g. In which dimensions is this theory superrenormalizable, in which renormalizable and in which unrenormalizable?