**Project** – Renormalizability of QED at one-loop

Show that QED is renormalizable at one-loop using renormalized perturbation theory, dimensional regularization ($d = 4 - \varepsilon$) and minimal subtraction. Proceed as follows.

*Renormalized perturbation theory:*
The renormalized QED Lagrangian reads

$$L = -\tfrac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + i Z_2 \bar{\psi}\, \partial\!\!\!/ \psi - Z_2 Z_m\, m\, \bar{\psi}\psi - e Z_e Z_2 \sqrt{Z_3} \bar{\psi} A\!\!\!/ \psi \ ,$$

where $\psi, F, m, e$ denote *renormalized* quantities (i.e. we drop the index $R$ used in class for notational ease). Define $Z_1 \equiv Z_e Z_2 \sqrt{Z_3}$. At one-loop we expand $Z_X$ (with $X = 1, 2, 3, m$) as

$$Z_X = 1 + \hbar\, \delta_X \quad , \quad X = 1, 2, 3, m \ .$$

The resulting Lagrangian for renormalized pertubation theory is

$$\begin{aligned} L \;=\; & -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \bar{\psi}\, \partial\!\!\!/ \psi - m\, \bar{\psi}\psi - e \bar{\psi} A\!\!\!/ \psi \\ & -\tfrac{1}{4} \hbar \delta_3 F^{\mu\nu} F_{\mu\nu} + i\hbar\delta_2 \bar{\psi}\, \partial\!\!\!/ \psi - \hbar(\delta_2 + \delta_m)\, m\, \bar{\psi}\psi - e\hbar\delta_1 \bar{\psi} A\!\!\!/ \psi \ . \end{aligned}$$

In renormalized perturbation theory, the $\delta_X$, called counterterms, are treated as additional interaction terms in the Lagrangian.

*Goal:*
The goal of the project consists in showing that the UV divergences that arise in three one-loop 1PI graphs are cancelled by a judicious choice of $\delta_X$, namely by picking

$$\delta_1 = \delta_2 = -\frac{e^2}{8\pi^2\, \varepsilon} \quad , \quad \delta_3 = -\frac{e^2}{6\pi^2\, \varepsilon} \quad , \quad \delta_m = -\frac{3e^2}{8\pi^2\, \varepsilon} \ . \tag{1}$$

The three one-loop graphs are: vacuum polarisation, electron self-energy, vertex.

*Feynman rules:*
The associated Feynman rules in momentum space are:

- photon propagator (in Feynman gauge) and fermion propagator:

$$i G^{tree}_{\mu\nu} = -i \frac{\eta_{\mu\nu}}{p^2 + i\epsilon} \quad , \quad i G^{tree} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}$$

- $-1$ for each fermion loop
- interactions:
  - vertex:

$$-ie\gamma^\mu$$

  - counterterm insertion on a fermion line:

$$i\hbar\left(p\!\!\!/\, \delta_2 - (\delta_m + \delta_2) m\right)$$

– counterterm insertion on a photon line (in Feynman gauge):

$$-i\hbar\,\delta_3\,p^2\,\eta_{\mu\nu}$$

– vertex counterterm:

$$-i\hbar\,\delta_1 e\gamma^\mu$$

*Propagators:*
The resulting one-loop corrected propagators are

- photon propagator (in Feynman graph):

$$iG_{\mu\nu} = iG^{tree}_{\mu\nu} + iG^{tree}_{\mu\alpha}\left(-ip^2\eta^{\alpha\beta}\,\left(e^2\,\Pi_2(p^2) + \hbar\delta_3\right)\right) iG^{tree}_{\beta\nu} + \mathcal{O}\left(e^4\right)$$

- fermion propagator:

$$iG = iG^{tree} + iG^{tree} i\left(\Sigma_2(\not p) + \hbar\not p\,\delta_2 - \hbar(\delta_m + \delta_2)m\right) iG^{tree} + \mathcal{O}\left(e^4\right)$$

*1PI graphs at 1-loop:*
*a) Vacuum polarisation:* The one-loop term $\Pi_2(p^2)$ arises by calculating the vacuum polarisation,

$$i\Pi_2^{\mu\nu} = -(-ie)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p-k)^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\mathrm{Tr}\left[\gamma^\mu(\not k-\not p+m)\gamma^\nu(\not k+m)\right] .$$

Dropping terms proportional to $p^\mu$ and $p^\nu$ yields

$$i\Pi_2^{\mu\nu} = -4e^2\int\frac{d^4k}{(2\pi)^4}\frac{[2k^\mu k^\nu + \eta^{\mu\nu}(-k^2+p\cdot k+m^2)]}{[(p-k)^2-m^2+i\epsilon][k^2-m^2+i\epsilon]} .$$

Introducing Feynman parameters, performing an appropriate shift of $k^\mu$ and dropping terms proportional to $p^\mu p^\nu$, $p^\mu$ and $p^\nu$, gives

$$\Pi_2^{\mu\nu} = -p^2\eta^{\mu\nu}\,e^2\,\Pi_2(p^2)$$

which, in dimensional regularization, becomes

$$\Pi_2(p^2) = \frac{8}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\mu^{4-d}\int_0^1 dx\,x(1-x)\left(\frac{1}{m^2-p^2x(1-x)}\right)^{2-\frac{d}{2}} .$$

Here we have used the relation (prove it!)

$$\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2-\Delta+i\epsilon)^n} = \frac{1}{d}\eta^{\mu\nu}\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-\Delta+i\epsilon)^n} \quad , \quad n\geq 2 . \tag{2}$$

*b) Electron self-energy:* The one-loop term $\Sigma_2(\not p)$ arises from the calculation of the electron self-energy,

$$i\Sigma_2(\not p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\,\frac{i(\not k+m)}{k^2-m^2+i\epsilon}\gamma_\mu\frac{(-i)}{(p-k)^2+i\epsilon} .$$

Introducing Feynman parameters, performing an appropriate shift of $k^\mu$ and using dimensional regularization, this becomes

$$\Sigma_2(\not p) = -2ie^2\mu^{4-d}\int_0^1 dx\ \left(x\not p - 2m\right)\int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-(1-x)(m^2-p^2x)+i\epsilon]^2}\ .$$

*c) Vertex:* The one-loop vertex 3-point function $-ieV^\mu(p,q_1)$ reads

$$-ieV^\mu(p,q_1) = (-ie)^3\int\frac{d^4k}{(2\pi)^4}\frac{-i}{(k-q_1)^2+i\epsilon}\gamma^\nu\ \frac{i(\not p+\not k+m)}{(p+k)^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not k+m)}{k^2-m^2+i\epsilon}\gamma_\nu\ .$$

Introducing Feynman parameters to rewrite the denominator, and performing an appropriate shift of $k^\mu$, show that the denominator becomes $(k^2-\Delta+i\epsilon)^3$ with

$$\Delta = -xyp^2+(x+y)^2m^2\ .$$

Now extract the term with two powers of $k$ in the numerator. This is the UV-divergent term (why?). Using dimensional regularization and (2), show that the UV-divergent part of $-ieV^\mu$ takes the form $V^\mu = F_1(p^2)\,\gamma^\mu$ with

$$F_1(p^2) = -2ie^2\mu^{4-d}\int_0^1 dx\int_0^x dy\int\frac{d^dk}{(2\pi)^d}\frac{(2-\frac{4}{d})\,k^2}{(k^2-\Delta+i\epsilon)^3}\ .$$

Using these results, derive the relations (1).

*Useful relations:*
*Gamma matrix algebra in d-dimensions:*
Using $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$ , $\eta^\mu{}_\mu = d$, prove the following relations:

$$\gamma^\mu\gamma_\mu = d\,\mathbb{I}\quad,\quad\gamma^\mu\not p\gamma_\mu = (2-d)\not p\quad,\quad\gamma^\mu\not p\not q\gamma_\mu = 4p\cdot q\ \mathbb{I}+(d-4)\not p\not q\ ,$$
$$\gamma^\nu\not k\gamma^\mu\not k\gamma_\nu = (d-2)\,k^2\,\gamma^\mu+(4-2d)\not k\,k^\mu\ .$$

Further relations ($d=4$): the trace of an odd number of $\gamma$'s vanishes, and

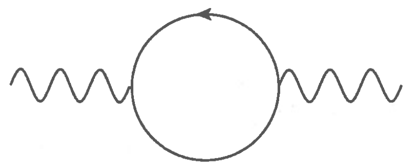
$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4\eta^{\mu\nu}\quad,\quad\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\ \gamma^\sigma\right) = 4\left(\eta^{\mu\nu}\ \eta^{\rho\sigma}+\eta^{\mu\sigma}\ \eta^{\nu\rho}-\eta^{\mu\rho}\ \eta^{\nu\sigma}\right)\ .$$

*Feynman parameters:*

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{[A+(B-A)x]^2}$$
$$\frac{1}{ABC} = \int_0^1 dx\int_0^x dy\,\frac{2}{[xA+yB+C(1-x-y)]^3}\ .$$

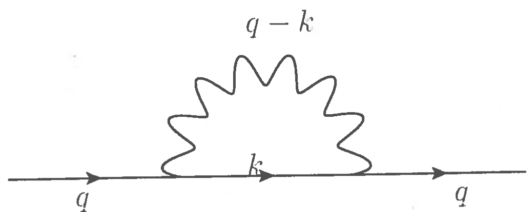
*Integrals:*

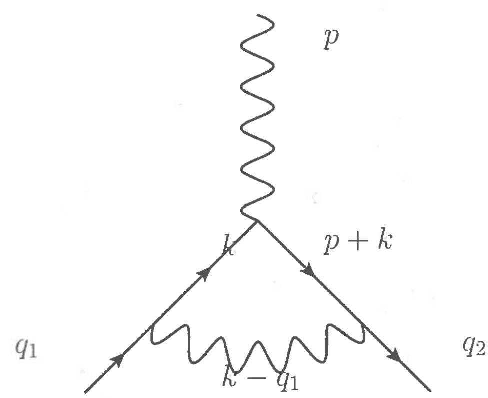
$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta+i\epsilon)^2} = \frac{i}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma\left(\frac{4-d}{2}\right)\ ,$$
$$\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-\Delta+i\epsilon)^2} = -\frac{d}{2}\frac{i}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-\frac{d}{2}}}\Gamma\left(\frac{2-d}{2}\right)\ ,$$
$$\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-\Delta+i\epsilon)^3} = \frac{d}{4}\frac{i}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma\left(\frac{4-d}{2}\right)\ .$$



$= \mathrm{i}\,\Pi_2^{\mu\nu}(q)$

**Electron self energy**




$= -\mathrm{i}\,eV^{\mu}_{1-loop}$