Project: give a detailed account, explaining each of the steps you use. Use LaTex.

Part 1 – The one-loop partition function of the $S^1/\mathbb{Z}_2$ orbifold

Compute the one-loop partition function of the $S^1/\mathbb{Z}_2$ orbifold of the free bosonic field theory

$$\mathcal{S} = \frac{1}{2\pi} \int d^2z \, \partial X \, \bar{\partial} X \ ,$$

with X compactified on a circle of radius R, i.e. $X \sim X + 2\pi R$.

Show that the resulting partition function is modular invariant.

Literature: section 8.4 of `Applied Conformal Field Theory`, https://arxiv.org/abs/hep-th/9108028

Part 2 – The partition function of the closed bosonic string compactified on a torus T^d

The partition function of the closed bosonic string compactified on a torus T^d is given by

$$Z(\tau, \bar{\tau}) = {\tau_2}^{-(24-d)/2} \, |\eta(\tau)|^{-48} \sum_{(p_L, p_R) \in \Gamma_{d,d}} \bar{q}^{\frac{1}{2}\, p_L^2} \, q^{\frac{1}{2} p_R^2} \quad , \quad \tau = \tau_1 + i\tau_2 \in \mathcal{H} \, , \tag{1}$$

where $|\eta(\tau)|^{-48}$ is the bosonic oscillator contribution, and ${\tau_2}^{-(24-d)/2}$ is the contribution from the transverse non-compact momenta. $\Gamma_{d,d}$ denotes an even, self-dual Lorentzian lattice, the Narain lattice, with

$$\begin{aligned} p_L &= \left(\sqrt{\frac{\alpha'}{2}}\, m_i + \frac{1}{\sqrt{2\alpha'}}\, g_{ij}\, n^j - \frac{1}{\sqrt{2\alpha'}}\, b_{ij}\, n^j \right) e^{*i} \\ p_R &= \left(\sqrt{\frac{\alpha'}{2}}\, m_i - \frac{1}{\sqrt{2\alpha'}}\, g_{ij}\, n^j - \frac{1}{\sqrt{2\alpha'}}\, b_{ij}\, n^j \right) e^{*i} \ , \end{aligned} \tag{2}$$

and $g_{ij} = e_i \cdot e_j$ the metric on the lattice $\Lambda = \{\sum_{i=1}^d n^i \, e_i | n^i \in \mathbb{Z}\}$.

The one-loop vacuum amplitude is then

$$\int_{\mathcal{F}} \frac{d^2\tau}{4\, \tau_2^2} \, Z(\tau, \bar{\tau}) \, , \tag{3}$$

where $\mathcal{F}$ denotes the fundamental domain of $SL(2, \mathbb{Z})$.

a) Begin by briefly reviewing and outlining the derivation of (1).

b) Show that $p_{L,R}^2$ are invariant under T-duality,

$$\mathbf{m} \leftrightarrow \mathbf{n} \quad , \quad \frac{1}{\alpha'} \, (\mathbf{g} + \mathbf{b}) \leftrightarrow \alpha' \, (\mathbf{g} + \mathbf{b})^{-1} \tag{4}$$

(Literature: section 2.4 of `Target Space Duality in String Theory`, arXiv:hep-th/9401139)

c) Show that (1) is invariant under modular transformations, i.e. under $\tau \mapsto \tau + 1$, $\tau \mapsto -1/\tau$. The latter requires peforming a Poisson resummation on the lattice.

(Literature:
chapters 9.6 and 10.4 of `Basic Concepts of String Theory` by Blumenhagen, Lüst, Theisen;
page 42 of `Mock Theta Functions`, https://arxiv.org/pdf/0807.4834.pdf;
section 7.4 of `Les Houches Lectures on Fields, Strings and Duality`, arXiv:hep-th/9703136)

d) Consider the one-loop partition function

$$Z_{T^d}(\tau, \bar{\tau}) = |\eta(\tau)|^{-2d} \sum_{(p_L, p_R) \in \Gamma_{d,d}} \bar{q}^{\frac{1}{2} p_L^2} \, q^{\frac{1}{2} p_R^2} \quad , \quad \tau = \tau_1 + i\tau_2 \in \mathcal{H} \, , \tag{5}$$

which is related to the one in (1) in an obvious way. Perform a Poisson resummation over m_i, to obtain an expression of the form

$$Z_{T^d}(\tau, \bar{\tau}) = \sum_{k,n \in \Lambda} e^{-S_{k,n}} \, . \tag{6}$$

Identify the form of $S_{k,n}$. This is the path-integral computation of the partition function.

(Literature: section 7.6 of `Les Houches Lectures on Fields, Strings and Duality`, arXiv:hep-th/9703136)