

Choose one of the following two projects:

**Project 1** – The one-loop partition function of the  $S^1/\mathbb{Z}_2$  orbifold

Compute the one-loop partition function of the  $S^1/\mathbb{Z}_2$  orbifold of the free bosonic field theory

$$\mathcal{S} = \frac{1}{2\pi} \int d^2z \partial X \bar{\partial} X ,$$

with  $X$  compactified on a circle of radius  $R$ , i.e.  $X \equiv X + 2\pi R$ .

Show that the resulting partition function is modular invariant.

Literature: section 8 of **Applied Conformal Field Theory**, <https://arxiv.org/abs/hep-th/9108028>

**Project 2** – The partition function of the closed bosonic string compactified on a torus  $T^d$

The partition function of the closed bosonic string compactified on a torus  $T^d$  is given by

$$Z(\tau, \bar{\tau}) = \tau_2^{-(24-d)/2} |\eta(\tau)|^{-48} \sum_{(p_L, p_R) \in \Gamma_{d,d}} \bar{q}^{\frac{1}{2} p_L^2} q^{\frac{1}{2} p_R^2} , \quad \tau = \tau_1 + i\tau_2 \in \mathcal{H} , \quad (1)$$

where  $|\eta(\tau)|^{-48}$  is the bosonic oscillator contribution, and  $\tau_2^{-(24-d)/2}$  is the contribution from the transverse non-compact momenta.  $\Gamma_{d,d}$  denotes an even, self-dual Lorentzian lattice, the Narain lattice, with

$$\begin{aligned} p_L &= \left( \sqrt{\frac{\alpha'}{2}} m_i + \frac{1}{\sqrt{2\alpha'}} g_{ij} n^j - \frac{1}{\sqrt{2\alpha'}} b_{ij} n^j \right) e^{*i} \\ p_R &= \left( \sqrt{\frac{\alpha'}{2}} m_i - \frac{1}{\sqrt{2\alpha'}} g_{ij} n^j - \frac{1}{\sqrt{2\alpha'}} b_{ij} n^j \right) e^{*i} , \end{aligned} \quad (2)$$

and  $g_{ij} = e_i \cdot e_j$  the metric on the lattice  $\Lambda = \{ \sum_{i=1}^d n^i e_i | n^i \in \mathbb{Z} \}$ .

The one-loop vacuum amplitude is then

$$\int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2^2} Z(\tau, \bar{\tau}) , \quad (3)$$

where  $\mathcal{F}$  denotes the fundamental domain of  $SL(2, \mathbb{Z})$ .

a) Begin by briefly reviewing and outlining the derivation of (1).

b) Show that  $p_{L,R}^2$  are invariant under T-duality,

$$\mathbf{m} \leftrightarrow \mathbf{n} \quad , \quad \frac{1}{\alpha'} (\mathbf{g} + \mathbf{b}) \leftrightarrow \alpha' (\mathbf{g} + \mathbf{b})^{-1} \quad (4)$$

(Literature: section 2.4 of **Target Space Duality in String Theory**, arXiv:hep-th/9401139)

c) Show that (1) is invariant under modular transformations, i.e. under  $\tau \mapsto \tau + 1$ ,  $\tau \mapsto -1/\tau$ . The latter requires performing a Poisson resummation on the lattice.

(Literature:

chapters 9.6 and 10.4 of **Basic Concepts of String Theory** by Blumenhagen, Lüst, Theisen; section 7.4 of **Les Houches Lectures on Fields, Strings and Duality**, arXiv:hep-th/9703136)

d) Consider the one-loop partition function

$$Z_{T^d}(\tau, \bar{\tau}) = |\eta(\tau)|^{-2d} \sum_{(p_L, p_R) \in \Gamma_{d,d}} \bar{q}^{\frac{1}{2} p_L^2} q^{\frac{1}{2} p_R^2} \quad , \quad \tau = \tau_1 + i\tau_2 \in \mathcal{H} \quad , \quad (5)$$

which is related to the one in (1) in an obvious way. Perform a Poisson resummation over  $m_i$ , to obtain an expression of the form

$$Z_{T^d}(\tau, \bar{\tau}) = \sum_{k,n \in \Lambda} e^{-S_{k,n}} \quad . \quad (6)$$

Identify the form of  $S_{k,n}$ . This is the path-integral computation of the partition function.

(Literature: section 7.6 of **Les Houches Lectures on Fields, Strings and Duality**, arXiv:hep-th/9703136)