Exercise 1 – Torus partition function of the closed bosonic string

The torus partition function of the closed bosonic string,

$$Z(\tau,\bar\tau) = {\tau_2}^{-12}\, |\eta(\tau)|^{-48} \quad , \quad \tau = \tau_1 + i\tau_2 \in \mathcal{H}\,, \tag{1}$$

encodes information about the degeneracy of states of the closed bosonic string. Determine the number of closed string states with mass $\alpha'\, M^2 = 20$ (you may use a computer-algebra program such as Mathematica, Mapple or WolframAlpha).

Exercise 2 – Closed bosonic string theory compactified on a circle

Consider the closed bosonic string with one space-time coordinate (say x^{25}) curled up into a circle of radius R. Let K and W denote the momentum and the winding numbers associated with x^{25}, respectively. The number operators associated with the right- and left-movers satisfy $N - \tilde N = K\, W$, and the mass formula reads

$$\alpha'\, M^2 = \alpha' \left[\left(\frac{K}{R}\right)^2 + \left(\frac{W\, R}{\alpha'}\right)^2 \right] + 2\left(N + \tilde N - 2\right) \ . \tag{2}$$

Show that in the sector of states with $K \neq 0$ and $W \neq 0$ there are no additional states (beyond those discussed in the lectures) that can ever become massless.

Exercise 3 – Closed bosonic string theory compactified on a two-torus with a constant Kalb-Ramond field

Assume that x^2 and x^3 are each compactified into a circle of radius R. The corresponding string coordinates are called X^r, with $r = 2, 3$. Moreover, there is a non-vanishing Kalb-Ramond field $B_{23} = b$, with b a dimensionless constant. All other components of $B_{\mu\nu}$ vanish. The worldsheet action for the X^r reads ($T = 1/(2\pi\alpha')$)

$$S = \frac{T}{2} \int d\tau d\sigma \left[\left(\dot X^r\right)^2 - (X'^r)^2 - \partial_\tau X^\mu\, \partial_\sigma X^\nu\, B_{\mu\nu} \right] \ . \tag{3}$$

a) Consider the following expansion for the zero mode part of the coordinates,

$$X^r = x^r(\tau) + 2W_r\, R\, \sigma \quad , \quad 0 \le \sigma \le \pi\,, \tag{4}$$

and compute the worldsheet action (3).

b) Define momenta canonical to x^r, compute the Hamiltonian, and show that it takes the form

$$H = \alpha' \left[\left(\pi\, p_2 + \frac{b\,R}{2\,\alpha'}\, W_3\right)^2 + \left(\pi\, p_3 - \frac{b\,R}{2\,\alpha'}\, W_2\right)^2 \right] + \frac{R^2}{\alpha'} \left(W_2^2 + W_3^2\right) \ . \tag{5}$$

Note that the quantization conditions on the momenta are $\pi\, p_r = K_r/R$.

c) While we have only looked explicitly at the zero modes, the oscillator expansion of the coordinates works just as before. Write the appropriate expansions for the coordinates $X^2(\tau,\sigma)$ and $X^3(\tau,\sigma)$.

d) What does the mass square operator look like, and what is the constraint on the number operator difference $N - \tilde N$?