Exercise 1 – Nilpotency of the bosonic string BRST charge

Verify that nilpotency of the bosonic string BRST charge Q requires $D = 26$, by showing that

$$[Q, T(w)] = \frac{(D-26)}{12}\,\partial^3 c(w)\ , \tag{1}$$

where $T(z) = T^X(z) + T^g(z)$, with

$$\begin{aligned} T^X(z) &= -\frac{1}{2} : \partial X(z) \cdot \partial X(z) : \\ T^g(z) &= : (\partial b(z))\, c(z) - 2\partial (b(c) c(z)) : \\ Q &= \frac{1}{2\pi i} \oint_{C_0} dz\ : c(z)\Big[T^X(z) + \frac{1}{2}\, T^g(z)\Big] :\ . \end{aligned} \tag{2}$$

Exercise 2 – The Veneziano amplitude

Show that the four-tachyon scattering amplitude in (Neumann) open string theory,

$$\mathcal{A} = \int_0^\infty d\tau\, \langle 0; -k_1 |\, V_C(0,0)\, V_D(\tau, 0) | 0; k_4\rangle\ , \tag{3}$$

equals the Euler beta function $B(a, b)$,

$$B(a,b) = \int_0^1 dx\, x^{a-1}\,(1-x)^{b-1} \tag{4}$$

with

$$a = 1 + 2\alpha'\, k_1 \cdot k_2 \quad , \quad b = 1 + 2\alpha'\, k_2 \cdot k_3\ . \tag{5}$$

All four tachyon states are on-shell, i.e. $\alpha'\, k\cdot k = 1$ for each of them (i.e. $\alpha' M^2 = -1$). The associated vertex operators are

$$\begin{aligned} V(\tau,\sigma) &= : \mathrm{e}^{ik\cdot X(\tau,\sigma)} : \quad , \\ V_C(0,0) &= : \mathrm{e}^{ik_2\cdot X(0,0)} : \quad , \\ V_D(\tau,0) &= : \mathrm{e}^{ik_3\cdot X(\tau,0)} : \quad . \end{aligned} \tag{6}$$

Proceed as follows:

1. Using the mode expansion (we set $l = \pi$ for convenience, and perform the Wick rotation $\tau \to i\tau$)

$$X^\mu(\tau,\sigma) = x^\mu + 2i\,\alpha'\, p^\mu\, \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n}\, e^{n\tau}\, \cos(n\sigma) \tag{7}$$

   and the Hausdorff formula

$$\mathrm{e}^A\, \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\dots} \tag{8}$$

   show that

$$V(\tau,\sigma)|0;k\rangle = \mathrm{e}^{ik_\mu x^\mu}\, \mathrm{e}^{-\tau(1+2\alpha'\, k\cdot p)}\, \mathrm{e}^{-\sqrt{2\alpha'}\, k_\nu \sum_{n<0} \frac{1}{n}\, \alpha_n^\nu\, e^{n\tau}\, \cos(n\sigma)}\, |0;k\rangle\ . \tag{9}$$

2. Next, using

$$p^\mu|0;k\rangle = k^\mu\,|0;k\rangle \quad , \quad \mathrm{e}^{ik_{3\mu}\,x^\mu}\,|0;k_4\rangle = |0;k_3+k_4\rangle\ , \tag{10}$$

show that (3) can be written as

$$\mathcal{A} = \int_0^\infty d\tau\,\mathrm{e}^{-(1+2\alpha'\,k_3\cdot k_4)\,\tau}\,\langle 0;-k_1-k_2|\,\mathrm{e}^{-\sqrt{2\alpha'}\,k_{2\mu}\sum_{n>0}\frac{1}{n}\alpha_n^\mu}\,\mathrm{e}^{-\sqrt{2\alpha'}\,k_{3\nu}\sum_{m<0}\frac{1}{m}\alpha_m^\nu\,\mathrm{e}^{m\tau}}\,|0;k_3+k_4\rangle\ . \tag{11}$$

3. Now interchange the order of the two oscillator operators in this expression by using the Hausdorff formula once again. To this end show that (for $\tau\neq 0$)

$$[k_{2\mu}\sum_{n>0}\frac{1}{n}\,\alpha_n^\mu\,,\,k_{3\nu}\sum_{m<0}\frac{1}{m}\,\alpha_m^\nu\,\mathrm{e}^{m\tau}] = k_2\cdot k_3\,\log\Big(1-\mathrm{e}^{-\tau}\Big)\ . \tag{12}$$

It follows that (11) can be written as

$$\mathcal{A} = \int_0^\infty d\tau\,\mathrm{e}^{-(1+2\alpha'\,k_3\cdot k_4)\,\tau}\,\Big(1-\mathrm{e}^{-\tau}\Big)^{2\alpha'\,k_2\cdot k_3}\,\langle 0;-k_1-k_2|\,0;k_3+k_4\rangle\ . \tag{13}$$

4. The orthogonality of states enforces momentum conservation, i.e.

$$\langle 0;-k_1-k_2|\,0;k_3+k_4\rangle = \delta(k_1+k_2+k_3+k_4)\ . \tag{14}$$

Use this to show that with the change of variable $x=\mathrm{e}^{-\tau}$, the amplitude (13) can be brought into the form

$$\mathcal{A} = \int_0^1 dx\,x^{2\alpha'\,k_1\cdot k_2}\,(1-x)^{2\alpha'\,k_2\cdot k_3}\ , \tag{15}$$

up to a proportionality factor that includes the delta function (14). The amplitude (15) can then be written as

$$\mathcal{A} = \int_0^1 dx\,x^{-\alpha's-2}\,(1-x)^{-\alpha't-2} \tag{16}$$

in terms of the Mandelstam variables $s=-(k_1+k_2)^2$ and $t=-(k_2+k_3)^2$. Setting $a=-\alpha's-1$ and $b=-\alpha't-1$, yields the integral representation of the Euler beta-function $B(a,b)$, which can be expressed in terms of Γ-functions as

$$\mathcal{A} = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\ . \tag{17}$$

The amplitude is thus symmetric under the exchange of s and t.

Using an appropriate representation of the Gamma function, show that (3) contains an infinite number of s-channel (or t-channel) poles, which can be interpreted as exchange processes of (an infinite number of) massive string excitations.