
Exercise 1 – Generating functions for partitions and string entropy

Let N be a fixed, positive integer. A partition of N is a set of positive integers that add up to N. The order of the elements in the set is immaterial. The number $N = 4$, for instance, has 5 partitions. The number of partitions for a given N is denoted by $p(N)$.

A generating function for the number of partitions $p(N)$ is given by

$$\prod_{N=1}^{\infty} \left(1 - x^N\right)^{-1} = \sum_{N=0}^{\infty} p(N)\, x^N \ . \tag{1}$$

To evaluate the left-hand side, each factor is expanded as an infinite Taylor series around $x = 0$.

a) Test this formula for $N \leq 4$ and explain in words why it works in general.

b) Find a generating function for unequal partitions $q(N)$ and test it for low values of N.
(For example, the partitions of $N = 3$ into unequal parts are 3 and $2 + 1$.)

c) Now consider the transverse number operator of the Neumann open string,

$$\hat{N} = \sum_{i=1}^{24} \sum_{n=1}^{\infty} \alpha_{-n}^i \, \alpha_n^i \ , \tag{2}$$

and compute

$$\mathrm{Tr}\, x^{\hat{N}} \ , \tag{3}$$

where the trace is over all the open string states which, as you recall, are given by

$$|\phi\rangle = \left(a_1^\dagger\right)^{n_1} \left(a_2^\dagger\right)^{n_2} \cdots \left(a_k^\dagger\right)^{n_k} \cdots |0\rangle \quad , \quad n_k = 0, 1, 2, \dots \ , \tag{4}$$

where we have suppressed the indices $i = 1, \dots, 24$. Show that

$$\mathrm{Tr}\, x^{\hat{N}} = [f(x)]^{-24} \quad , \quad f(x) = \prod_{N=1}^{\infty} \left(1 - x^N\right) \ . \tag{5}$$

d) Let us denote the total number of Neumann open string states with mass $\alpha' M^2 = N - 1$ by d_N. From the above we infer that d_N can be extracted from the generating function

$$\mathrm{Tr}\, z^{\hat{N}} = \sum_{N=0}^{\infty} d_N \, z^N \tag{6}$$

via the contour integral

$$d_N = \frac{1}{2\pi i} \oint dz \, \frac{[f(z)]^{-24}}{z^{N+1}} \ . \tag{7}$$

This integral can be estimated for large N by a saddle point evaluation. To this end show that $f(x)$ can be written as

$$f(x) = \exp\left(- \sum_{n=1}^{\infty} \frac{x^n}{n(1 - x^n)} \right) \ . \tag{8}$$

Next, show that for $x \to 1$ this can be approximated by

$$f(x) \approx \exp\left(-\frac{\pi^2}{6(1-x)}\right) . \tag{9}$$

Finally show that for large N the function $[f(z)]^{-24}/z^{N+1}$ has an extremum near $z = 1$, and that this function takes the value $\exp[4\pi\sqrt{N+1}]$ there. Hence, using a saddle point approximation, we conclude that

$$d_N \approx \mathrm{e}^{4\pi\sqrt{N}} \quad \text{as} \quad N \to \infty . \tag{10}$$

It follows that the microscopic entropy for fixed and large N is given by

$$\mathcal{S}_{\text{micro}} = k_B \log d_N \approx k_B\, 4\pi\sqrt{N} \sim M\, l_s . \tag{11}$$

Therefore, the free string entropy depends linearly on the mass M. Since we may heuristically estimate the length of a string with mass M to be $M \sim T\,L \sim L/\alpha'$, we see that the string entropy is an extensive quantity.

Exercise 2 – Long strings are entropically favored

Consider Neumann open string states with zero spatial momentum, so that their energy is given by $\alpha'\, E^2 = N - 1$. How many times is the number of available states increased when a string with $N = 9$ is formed from two strings, each of which has the same energy? What is the change of entropy? (Consider only tranverse excitations. A little help: for $N = 9$, $d_N = 143184000$.)

Exercise 3 – Polarization tensor/Vertex operators

a) Consider the closed string state

$$|\phi\rangle = \zeta_{\mu\nu}\, \alpha^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1}\, |0; k\rangle ,$$

where $\zeta_{\mu\nu}$ denotes the so-called polarization tensor. What do the Virasoro constraints on physical states,

$$L_m|\phi\rangle = \tilde{L}_m|\phi\rangle = 0 \quad , \quad m > 0 ,$$

imply for $\zeta_{\mu\nu}$?

b) The vertex operator associated to the above state is

$$V(z, \bar{z}) = \zeta_{\mu\nu}\ :\partial X^{\mu}(z)\, \bar{\partial} X^{\nu}(\bar{z})\, e^{ik\cdot X(z,\bar{z})}: \ .$$

Compute its OPE with the energy-momentum tensor $T(z)$ and with $\bar{T}(\bar{z})$, and demand V to be a primary field with conformal weight $(h, \bar{h}) = (1, 1)$. What do these conditions imply for $k^2 = k \cdot k$ and for $\zeta_{\mu\nu}$?