Exercise 1 – The Veneziano amplitude

Show that the four-tachyon scattering amplitude in (Neumann) open string theory,

$$\mathcal{A} = \int_{-\infty}^{0} d\tau \, \langle -k_1 | \, V_C(0,0) \, V_D(\tau,0) | k_4 \rangle \; , \tag{1}$$

equals the Euler beta function $B(a,b)$,

$$B(a,b) = \int_0^1 dx \, x^{a-1} \, (1-x)^{b-1} \tag{2}$$

with

$$a = 1 + 2\alpha' \, k_1 \cdot k_2 \quad , \quad b = 1 + 2\alpha' \, k_2 \cdot k_3 \; . \tag{3}$$

All four tachyon states are on-shell, i.e. $\alpha' \, k \cdot k = 1$ for each of them (i.e. $\alpha' M^2 = -1$). The associated vertex operators are

$$\begin{aligned} V(\tau,\sigma) &= \; : \mathrm{e}^{ik\cdot X(\tau,\sigma)} : \quad , \\ V_C(0,0) &= \; : \mathrm{e}^{ik_2\cdot X(0,0)} : \quad , \\ V_D(\tau,0) &= \; : \mathrm{e}^{ik_3\cdot X(\tau,0)} : \quad . \end{aligned} \tag{4}$$

Proceed as follows:

1. Using the mode expansion (and perform the Wick rotation $\tau \to -i\tau$)

$$X^\mu(\tau,\sigma) = x^\mu - 2i \, \alpha' \, p^\mu \, \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{a_n^\mu}{n} \, \mathrm{e}^{-n\tau} \, \cos(n\sigma) \tag{5}$$

   and the Hausdorff formula

$$\mathrm{e}^A \, \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\dots} \tag{6}$$

   show that

$$V(\tau,\sigma)|k\rangle = \mathrm{e}^{ik_\mu x^\mu} \, \mathrm{e}^{\tau(1+2\alpha' \, k\cdot p)} \, \mathrm{e}^{-\sqrt{2\alpha'} \, k_\nu \sum_{n<0} \frac{1}{n} a_n^\nu \, \mathrm{e}^{-n\tau} \, \cos(n\sigma)} \, |k\rangle \; . \tag{7}$$

2. Next, using

$$p^\mu |k\rangle = k^\mu \, |k\rangle \quad , \quad \mathrm{e}^{ik_{3\mu} \, x^\mu} \, |k_4\rangle = |k_3 + k_4\rangle \; , \tag{8}$$

   show that (1) can be written as

$$\mathcal{A} = \int_{-\infty}^{0} d\tau \, \mathrm{e}^{(1+2\alpha' \, k_3\cdot k_4)\,\tau} \, \langle -k_1 - k_2 | \, \mathrm{e}^{-\sqrt{2\alpha'} \, k_{2\mu} \sum_{n>0} \frac{1}{n} a_n^\mu} \, \mathrm{e}^{-\sqrt{2\alpha'} \, k_{3\nu} \sum_{m<0} \frac{1}{m} a_m^\nu \, \mathrm{e}^{-m\tau}} \, |k_3 + k_4\rangle \; . \tag{9}$$

3. Now interchange the order of the two oscillator operators in this expression by using the Hausdorff formula once again. To this end show that for $-\infty < \tau < 0$,

$$[k_{2\mu} \sum_{n>0} \frac{1}{n} \, a_n^\mu \, , \, k_{3\nu} \sum_{m<0} \frac{1}{m} \, a_m^\nu \, \mathrm{e}^{-m\tau}] = k_2 \cdot k_3 \, \log\left(1 - \mathrm{e}^\tau\right) \; . \tag{10}$$

   It follows that (9) can be written as

$$\mathcal{A} = \int_{-\infty}^{0} d\tau \, \mathrm{e}^{(1+2\alpha' \, k_3\cdot k_4)\,\tau} \, \left(1 - \mathrm{e}^\tau\right)^{2\alpha' \, k_2\cdot k_3} \, \langle -k_1 - k_2 | \, k_3 + k_4 \rangle \; . \tag{11}$$

4. The orthogonality of states enforces momentum conservation, i.e.
$$\langle -k_1 - k_2 | \, k_3 + k_4 \rangle = \delta(k_1 + k_2 + k_3 + k_4) \; . \tag{12}$$
Use this to show that with the change of variable $x = \mathrm{e}^{\tau}$, the amplitude (11) can be brought into the form
$$\mathcal{A} = \int_0^1 dx \, x^{2\alpha' \, k_1 \cdot k_2} \, (1-x)^{2\alpha' \, k_2 \cdot k_3} \; , \tag{13}$$
up to a proportionality factor that includes the delta function (12). The amplitude (13) can then be written as
$$\mathcal{A} = \int_0^1 dx \, x^{-\alpha' s - 2} \, (1-x)^{-\alpha' t - 2} \tag{14}$$
in terms of the Mandelstam variables $s = -(k_1 + k_2)^2$ and $t = -(k_2 + k_3)^2$. Setting $a = -\alpha' s - 1$ and $b = -\alpha' t - 1$, yields the integral representation of the Euler beta-function $B(a, b)$, which can be expressed in terms of Γ-functions as
$$\mathcal{A} = \frac{\Gamma(a) \, \Gamma(b)}{\Gamma(a+b)} \; . \tag{15}$$
The amplitude is thus symmetric under the exchange of s and t.

Using an appropriate representation of the Gamma function, show that (1) contains an infinite number of s-channel (or t-channel) poles, which can be interpreted as exchange processes of (an infinite number of) massive string excitations.

Exercise 2 – Radial and normal ordering

Consider chiral primary fields ϕ and χ of conformal weight h_ϕ and h_χ, respectively. Their mode expansion is $\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n \, z^{-n-h_\phi}$ and $\chi(w) = \sum_{m \in \mathbb{Z}} \chi_n \, w^{-m-h_\chi}$, respectively.

Consider their radially ordered OPE, i.e. $R\left(\phi(z) \, \chi(w)\right) = \{ \text{ singular terms } \} + \Sigma(z, w)$, where $\Sigma(z, w)$ denotes the regular part of the OPE, i.e. $\Sigma(z, w) = \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \, \Sigma_n(w)$.
Show that the modes B_k in the mode expansion $\Sigma_0(w) = \sum_{k \in \mathbb{Z}} B_k \, w^{-k-h_\phi-h_\chi}$ are given in terms of normal ordered products of the modes ϕ_n, χ_m,
$$B_k = \sum_{m \in \mathbb{Z}} \, : \phi_m \, \chi_{k-m} : \; .$$
Hint: apply $\frac{1}{2\pi i} \oint_{C_w} dz \, (z-w)^{-1}$ to both sides of the OPE of $R\left(\phi(z) \, \chi(w)\right)$, and reexpress the integral $\oint_{C_w}$ in terms of integrals $\oint_{|z|>|w|}$ and $\oint_{|w|>|z|}$.

Exercise 3 – Nilpotency of the bosonic string BRST charge

Verify that nilpotency of the bosonic string BRST charge Q requires $D = 26$, by showing that
$$[Q, T(w)] = \frac{(D-26)}{12} \, \partial^3 c(w) \; ,$$
where $T(z) = T^X(z) + T^g(z)$, with
$$\begin{aligned} T^X(z) &= -\frac{1}{2} : \partial X(z) \cdot \partial X(z) : \\ T^g(z) &= : (\partial b(z)) \, c(z) - 2 \partial (b(c) c(z)) : \\ Q &= \frac{1}{2\pi i} \oint_{C_0} dz \; : c(z) \Big[T^X(z) + \frac{1}{2} \, T^g(z) \Big] : \; . \end{aligned}$$