Exercise 1 – p-brane action

Consider a p-brane moving in a spacetime (M, g), which we take to be d-dimensional Minkowski spacetime $(\mathbb{R}^{1,d-1}, \eta)$. The action for this p-brane is given by

$$S = -\frac{T_p}{2} \int_\Sigma d^{p+1}\sigma \sqrt{-h}\, h^{\alpha\beta}\, \eta_{\mu\nu}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu + \Lambda_p \int_\Sigma d^{p+1}\sigma \sqrt{-h}\ .$$

Here, the embedding $X : (\Sigma, h_{\alpha\beta}) \to (M, \eta_{\mu\nu})$ denotes a smooth map between two semi-Riemannian manifolds, the $(p+1)$-dimensional world volume of the p-brane and d-dimensional Minkowski spacetime, and $h \equiv \det h_{\alpha\beta}$.
Investigate whether the equation of motion for the world-volume metric $h_{\alpha\beta}$ is solved by equating the world-volume metric with the induced metric $X^*\eta$. Show that this requires a non-vanishing cosmological constant Λ_p for $p \neq 1$.

Exercise 2 – Weyl rescalings in two dimensions

Consider a two-dimensional semi-Riemannian manifold $(M, h_{\alpha\beta})$.

a) Show that under local Weyl rescalings of the metric, $h_{\alpha\beta} \to e^{2\Lambda(\sigma)}\, h_{\alpha\beta}$, the quantity $\sqrt{-h}\,\mathcal{R}$ transforms as

$$\sqrt{-h}\,\mathcal{R} \to \sqrt{-h}\,\left(\mathcal{R} - 2\nabla^2\Lambda\right)\ . \tag{1}$$

Here, $h \equiv \det h_{\alpha\beta}$, and $\mathcal{R}$ denotes the Ricci scalar ($\mathcal{R} = h^{\alpha\beta}\,\mathrm{Ric}_{\alpha\beta}\,,\ \mathrm{Ric}_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$).

b) Conclude that one may add a term $\int_\Sigma d^2\sigma \sqrt{-h}\,\mathcal{R}$ to the world-sheet action of a closed string while maintaining invariance under local Weyl rescalings.

c) Use the result (1) to show that locally, every metric of signature $(-1, 1)$ can be brought into the form $\eta = \mathrm{diag}(-1, 1)$ by Weyl rescalings and appropriate choice of local coordinates.

Exercise 3 – Witt algebra

In the conformal gauge, and in light-cone coordinates $\sigma_\pm = \tau \pm \sigma$, the world-sheet energy-momentum tensor $T_{\alpha\beta}$ has components given by $T_{+-} = T_{-+} = 0$, $T_{\pm\pm} = -\partial_\pm X^\mu\, \partial_\pm X^\nu\, \eta_{\mu\nu}$, where $\partial_\pm = \partial_{\sigma^\pm}$.

Consider the closed string.

a) Using the equal time Poisson brackets $\{\cdot,\cdot\}$,

$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = 0\ ,\ \{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = 2\pi\alpha'\,\eta^{\mu\nu}\,\delta(\sigma - \sigma')\ ,$$

calculate

$$\{T_{\pm\pm}(\sigma,\tau), X^\mu(\sigma',\tau)\}\ .$$

b) On-shell, $T_{--} = T_{--}(\sigma^-)$, $T_{++} = T_{++}(\sigma^+)$.
Define charges

$$L_{\epsilon^-} \equiv -\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, \epsilon^-(\sigma^-)\, T_{--}(\sigma^-) \ ,$$

with $\epsilon^-(\sigma^-)$ a periodic function in σ. Show that L_{ϵ^-} is conserved and

$$\{L_{\epsilon^-}, X^\mu(\sigma,\tau)\} = -\alpha'\, \mathcal{L}_{\epsilon^-} X^\mu(\sigma,\tau) \ ,$$

where $\mathcal{L}$ denotes the Lie derivative.
Using this result, and Fourier decomposing

$$T_{--}(\sigma^-) = -\alpha' \sum_{n\in\mathbb{Z}} L_m\, e^{-im\sigma^-} \ ,$$

show that

$$\{L_m, X^\mu(\sigma,\tau)\} = T_m X^\mu(\sigma,\tau) \ ,$$

where the vector field T_m is an element of the Lie algebra

$$[T_m, T_n] = i(m-n)\, T_{m+n} \ .$$

c) Show that the generators L_m satisfy the Witt algebra

$$\{L_m, L_n\} = -i(m-n)L_{m+n}$$

Verify that these commutation relations satisfy the Jacobi identity.