String Theory **Assignment Sheet 2** **Due: March 16, 2017**

Exercise 1 – Open string spectrum

Construct the spectrum of NN open strings in light-cone gauge for level $N = 3$. How many states are there?

Exercise 2 – Virasoro algebra

Let L_n denote the normal ordered operators arising in light-cone quantization,

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} :\alpha^i_{n-m}\,\alpha^i_m: \quad , \quad i = 1, \ldots, D-2 \quad , \quad n \in \mathbb{Z} \,. \tag{1}$$

a) Using the commutator relation $[A, BC] = [A, B]C + B[A, C]$ show that

$$[\alpha^i_m, L_n] = m\,\alpha^i_{m+n} \,. \tag{2}$$

b) Using (2), show that the L_n satisfied the following, centrally extended algebra, called Virasoro algebra,

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}\,m\,(m^2-1)\,\delta_{m+n,0} \quad , \tag{3}$$

where $[c, L_n] = 0$. Verify that the central charge c arises because of normal ordering, and determine its value.

c) When $c = 0$, the algebra is called Witt algebra. Verify that the vector fields

$$L_n = i e^{in\sigma^\pm} \frac{\partial}{\partial \sigma^\pm} \quad , \quad n \in \mathbb{Z} \tag{4}$$

give a representation of the Witt algebra.

Exercise 3 – Analytic continuation of the zeta function $\zeta(s)$

Consider the gamma function $\Gamma(s) = \int_0^\infty dt\,\mathrm{e}^{-t}\,t^{s-1}$, $s \in \mathbb{C}$. It is absolutely convergent for $\Re(s) > 0$. Let $t \to n\,t$ in this integral, and use the resulting equation to prove that

$$\Gamma(s)\,\zeta(s) = \int_0^\infty dt\,\frac{t^{s-1}}{\mathrm{e}^t - 1} \quad , \quad \Re(s) > 1 \,, \tag{5}$$

where $\zeta(s) = \sum_{n=1}^\infty n^{-s}$, $\Re(s) > 1$. Show that for $\Re(s) > 1$

$$\begin{aligned} \Gamma(s)\,\zeta(s) \;&=\; \int_0^1 dt\,t^{s-1}\left(\frac{1}{\mathrm{e}^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} \\ &\quad + \int_1^\infty dt\,\frac{t^{s-1}}{\mathrm{e}^t-1} \,. \end{aligned} \tag{6}$$

Explain why the first integral on the right-hand side above is well behaved for $\Re(s) > -2$.
The right-hand side defines the analytic continuation of the left-hand side to $\Re(s) > -2$. Using that $\Gamma(s)$ has a simple pole at $s = -1$ with residue -1, show that $\zeta(-1) = -1/12$, a celebrated result used in light-cone quantization.