Exercise 1 – Radial and normal ordering

Consider chiral primary fields ϕ and χ of conformal weight h_ϕ and h_χ, respectively. Their mode expansion is $\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n \, z^{-n-h_\phi}$ and $\chi(w) = \sum_{m\in\mathbb{Z}} \chi_n \, w^{-m-h_\chi}$, respectively.

Consider their radially ordered OPE (with $|z| > |w|$), i.e. $\phi(z)\,\chi(w) = \{$ singular terms $\} + \Sigma(z,w)$, where $\Sigma(z,w)$ denotes the regular part of the OPE, i.e. $\Sigma(z,w) = \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \, \Sigma_n(w)$.
Show that the modes B_k in the mode expansion $\Sigma_0(w) = \sum_{k\in\mathbb{Z}} B_k \, w^{-k-h_\phi-h_\chi}$ are given in terms of normal ordered products of the modes ϕ_n, χ_m.
Hint: apply $\frac{1}{2\pi i} \oint_{C_w} dz \, (z-w)^{-1}$ to both sides of the OPE of $\phi(z)\,\chi(w)$, and reexpress the integral $\oint_{C_w}$ in terms of integrals $\oint_{|z|>|w|}$ and $\oint_{|w|>|z|}$.

Exercise 2 – Nilpotency of the bosonic string BRST charge

Verify that nilpotency of the bosonic string BRST charge Q requires $D = 26$, by showing that

$$[Q, T(w)] = \frac{(D-26)}{12} \, \partial^3 c(w) \; ,$$

where $T(z) = T^X(z) + T^g(z)$, with

$$\begin{aligned} T^X(z) &= -\frac{1}{2} : \partial X(z) \cdot \partial X(z) : \\ T^g(z) &= : (\partial b(z)) \, c(z) - 2\partial(b(z)c(z)) : \\ Q &= \frac{1}{2\pi i} \oint_{C_0} dz \; : c(z) \Big[T^X(z) + \frac{1}{2} T^g(z) \Big] : \; . \end{aligned}$$

Exercise 3 – The Veneziano amplitude

Show that the four-tachyon scattering amplitude in (NN) open string theory,

$$\mathcal{A} = \int_0^\infty d\tau \, \langle -k_1 | \, V_C(0,0) \, V_D(\tau,0) | k_4 \rangle \; ,$$

equals the Euler beta function $B(a,b)$,

$$B(a,b) = \int_0^1 dy \, y^{a-1} \, (1-y)^{b-1}$$

with

$$a = 1 + 2\alpha' \, k_1 \cdot k_2 \quad , \quad b = 1 + 2\alpha' \, k_2 \cdot k_3 \; .$$

All four tachyon states are on-shell, i.e. $\alpha' \, k \cdot k = 1$ for each of them (i.e. $\alpha' M^2 = -1$). The associated vertex operators are

$$\begin{aligned} V_C(0,0) &= \; : \mathrm{e}^{ik_2 \cdot X(0,0)} : \quad , \\ V_D(\tau,0) &= \; : \mathrm{e}^{ik_3 \cdot X(\tau,0)} : \quad , \end{aligned}$$

with mode expansion (we perform the Wick rotation $\tau \to i\tau$)

$$X^\mu(\tau,\sigma) = x^\mu + 2i \, \alpha' \, p^\mu \, \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{a_n^\mu}{n} \, e^{n\tau} \, \cos(n\sigma) \; .$$

Hint: first show

$$V(\tau,0)|k\rangle = \mathrm{e}^{ik_\mu x^\mu} \, \mathrm{e}^{-\tau(1+2\alpha' \, k\cdot p)} \, \mathrm{e}^{-\sqrt{2\alpha'} \, k_\nu \sum_{n<0} \frac{1}{n} \, a_n^\nu \, \mathrm{e}^{n\tau}} \, |k\rangle \; .$$