Exercise 1 – Generating functions for partitions and string entropy

Let N be a fixed, positive integer. A partition of N is a set of positive integers that add up to N. The order of the elements in the set is immaterial. The number $N = 4$, for instance, has 5 partitions. The number of partitions for a given N is denoted by $p(N)$.

The generating function for the number of partitions $p(N)$ is given by

$$\prod_{N=1}^{\infty} \left(1 - x^N\right)^{-1} = \sum_{N=0}^{\infty} p(N)\, x^N \ .$$

a) Test this formula for $N \leq 4$ and explain why it works in general.

b) Find a generating function for unequal partitions $q(N)$ and test it for low values of N. (For example, the partitions of $N = 3$ into unequal parts are 3 and $2 + 1$.)

c) Now consider the transverse number operator of the Neumann open string,

$$\hat{N} = \sum_{i=1}^{24} \sum_{n=1}^{\infty} \alpha^i_{-n}\, \alpha^i_n \ ,$$

and compute

$$\mathrm{Tr}\, x^{\hat{N}} \ ,$$

where the trace is over all the open string states which, as you recall, are given by

$$|\phi\rangle = \left(a_1^\dagger\right)^{n_1} \left(a_2^\dagger\right)^{n_2} \cdots \left(a_k^\dagger\right)^{n_k} \cdots |0\rangle \quad , \quad n_k = 0, 1, 2, \ldots \ ,$$

where we have suppressed the indices $i = 1, \ldots, 24$. Show that

$$\mathrm{Tr}\, x^{\hat{N}} = [f(x)]^{-24} \quad , \quad f(x) = \prod_{N=1}^{\infty} \left(1 - x^N\right) \ .$$

d) Let us denote the total number of Neumann open string states with mass $\alpha' M^2 = N - 1$ by d_N. From the above we infer that d_N can be extracted from the generating function

$$\mathrm{Tr}\, z^{\hat{N}} = \sum_{N=0}^{\infty} d_N\, z^N$$

via the contour integral

$$d_N = \frac{1}{2\pi i} \oint dz\, \frac{[f(z)]^{-24}}{z^{N+1}} \ .$$

This integral can be estimated for large N by a saddle point evaluation. To this end show that $f(x)$ can be written as

$$f(x) = \exp\left(- \sum_{n=1}^{\infty} \frac{x^n}{n(1 - x^n)} \right) \ .$$

Next, show that for $x \to 1$ this can be approximated by

$$f(x) \approx \exp\left(-\frac{\pi^2}{6(1-x)}\right) .$$

Finally show that for large N the function $[f(z)]^{-24}/z^{N+1}$ has an extremum near $z = 1$, and that this function takes the value $\exp[4\pi\sqrt{N+1}]$ there. Hence, using a saddle point approximation, we conclude that

$$d_N \approx \mathrm{e}^{4\pi\sqrt{N}} \quad \text{as} \quad N \to \infty .$$

It follows that the microscopic entropy for fixed and large N is given by

$$\mathcal{S}_{\text{micro}} = k_B \log d_N \approx k_B\, 4\pi\sqrt{N} \sim M\, l_s .$$

Therefore, the free string entropy depends linearly on the mass M. Since we may heuristically estimate the length of a string with mass M to be $M \sim T\,L \sim L/\alpha'$, we see that the string entropy is an extensive quantity.

Exercise 2 – Operator-state correspondence and correlation functions

a) Consider a string state $|\psi\rangle$ of the form

$$|\psi\rangle = \left(\frac{\alpha'}{2}\right)^{(r+s)/2} \prod_{c=1}^{r}(-n_c-1)! \prod_{d=1}^{s}(-m_s-1)!\; a_{n_1}^{\mu_1}\cdots a_{n_r}^{\mu_r}\,\tilde{a}_{m_1}^{\nu_1}\cdots\tilde{a}_{m_s}^{\nu_s}\,|k\rangle$$

with $n_c \leq -1$ and $m_d \leq -1$. Show that the associated operator is given by

$$V_\psi(z,\bar{z}) =: \partial^{-n_1-1} J^{\mu_1}(z)\cdots\partial^{-n_r-1} J^{\mu_r}(z)\,\bar{\partial}^{-m_1-1}\tilde{J}^{\nu_1}(\bar{z})\cdots\partial^{-m_s-1}\tilde{J}^{\nu_s}(\bar{z})\, e^{ik\cdot X(z,\bar{z})} :$$

b) Let $V_k(z,\bar{z}) =: e^{ik\cdot X(z,\bar{z})} :$. Show that

$$\langle 0|\prod_{i=1}^{3} V_{k_i}(z_i,\bar{z}_i)\,|0\rangle = \prod_{i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j}\,\delta^{(26)}\left(\sum_{i=1}^{3}k_i\right) ,$$

where $|z_1| > |z_2| > |z_3|$.

c) Show that under Möbius transformations

$$z \mapsto \gamma(z) = \frac{az+b}{cz+d} \quad , \quad z \in \mathbb{C} \quad , \quad a,b,c,d \in \mathbb{C} \quad , \quad ad-bc = 1 ,$$

we have

$$\langle 0|\prod_{i=1}^{3} V_{k_i}(z_i,\bar{z}_i)\,|0\rangle \mapsto \left(\prod_{i=1}^{3}\left|\frac{d\gamma(z_i)}{dz_i}\right|^{-\alpha' k_i^2/2}\right)\langle 0|\prod_{i=1}^{3} V_{k_i}(z_i,\bar{z}_i)\,|0\rangle .$$

d) Compute $\langle 0|T(z)\,T(w)|0\rangle$ with $|z| > |w|$, where $T(z) = \alpha' \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}$.