Exercise 1 – Open string mode expansion

Consider an open string $X(\tau, \sigma)$ with $\sigma \in [0, \pi]$.

a) Define $\hat{X}(\tau, \sigma)$ by

$$\hat{X}(\tau, \sigma) = \begin{cases} X(\tau, \sigma) \quad , \quad 0 \leq \sigma \leq \pi \\ X(\tau, -\sigma) \quad , \quad -\pi \leq \sigma < 0 \end{cases}$$

Show that the extension of $\hat{X}(\tau, \sigma)$ to $\sigma \in \mathbb{R}$ as a smooth periodic function with period 2π requires Neumann-Neumann (NN) boundary conditions for $X(\tau, \sigma)$ at $\sigma = 0, \pi$. Use this to obtain the mode expansion for the NN open string: consider first the mode expansion of $\partial_\pm \hat{X}$, then integrate this to obtain $X(\tau, \sigma)$ subject to NN boundary conditions.

b) Redo the procedure for Dirichlet-Dirichlet (DD) boundary condtions, by suitably defining $\hat{X}(\tau, \sigma)$ so as to incorporate DD boundary conditions for $X(\tau, \sigma)$. Then, integrate the solution for $\partial_\pm \hat{X}$ to obtain $X(\tau, \sigma)$ subject to DD boundary conditions.

c) Similarly, find the mode expansion for the DN open string and for the ND open string.

Exercise 2 – Virasoro algebra

Let L_n denote the normal ordered operators arising in light-cone quantization,

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha^i_{n-m} \, \alpha^i_m : \quad , \quad i = 1, \dots, D-2 \quad , \quad n \in \mathbb{Z} \, . \tag{1}$$

a) Using the commutator relation $[A, BC] = [A, B]C + B[A, C]$ show that

$$[\alpha^i_m, L_n] = m \, \alpha^i_{m+n} \, . \tag{2}$$

b) Using (2), show that the L_n satisfied the following, centrally extended algebra, called Virasoro algebra,

$$[L_m, L_n] = (m - n) \, L_{m+n} + \frac{c}{12} \, m \, (m^2 - 1) \, \delta_{m+n,0} \quad , \tag{3}$$

where $[c, L_n] = 0$. Verify that the central charge c arises because of normal ordering, and determine its value.

Exercise 3 – Analytic continuation of the zeta function $\zeta(s)$

Consider the gamma function $\Gamma(s) = \int_0^\infty dt \, \mathrm{e}^{-t} \, t^{s-1}$, $s \in \mathbb{C}$. It is absolutely convergent for $\Re(s) > 0$. Let $t \to n \, t$ in this integral, and use the resulting equation to prove that

$$\Gamma(s) \, \zeta(s) = \int_0^\infty dt \, \frac{t^{s-1}}{\mathrm{e}^t - 1} \quad , \quad \Re(s) > 1 \, , \tag{4}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $\Re(s) > 1$. Show that for $\Re(s) > 1$

$$\begin{aligned} \Gamma(s)\,\zeta(s) \quad &= \int_0^1 dt\, t^{s-1} \left(\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} \\ &+ \int_1^{\infty} dt\, \frac{t^{s-1}}{\mathrm{e}^t - 1} \,. \end{aligned} \tag{5}$$

Explain why the first integral on the right-hand side above is well behaved for $\Re(s) > -2$.
The right-hand side defines the analytic continuation of the left-hand side to $\Re(s) > -2$. Using that $\Gamma(s)$ has a simple pole at $s = -1$ with residue -1, show that $\zeta(-1) = -1/12$, a celebrated result used in light-cone quantization.