String Theory **Assignment Sheet 1** **Due: March 9, 2017**

Exercise 1 – Euler-Lagrange equations for a Lagrangian mechanical system

A Lagrangian mechanical system is a pair (M, L) consisting of a smooth manifold M (called configuration space, with $\dim M = n$) and a smooth function L (the Lagrangian function) on TM. Consider the action of a smooth curve $\gamma : [a, b] \to M$,

$$S(\gamma) = \int_a^b dt\, L(\gamma'(t)) \; . \tag{1}$$

A motion of the system is defined to be a critical point of S under smooth variations of γ with fixed endpoints (stationary action principle). Show that motions of the system are the solutions of the following system of ordinary differential equations (called Euler-Lagrange equations, or simply equations of motions)

$$\frac{\partial L}{\partial q^i}(\gamma') - \frac{d}{dt}\frac{\partial L}{\partial v^i}(\gamma') \quad , \quad i = 1, \ldots, n \; , \tag{2}$$

where $(q, v) = (q^1, \ldots, q^n, v^1, \ldots\, v^n)$ are local coordinates on an open subset of TM.

Exercise 2 – Relativistic point particle action

Consider a relativistic particle with mass m in d-dimensional Minkowski spacetime, a pseudo-Riemannian manifold $(M, g = \eta)$. Consider the point particle action

$$S = \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau \left(e^{-1} g(\dot{x}, \dot{x}) - e m^2 c^2\right) . \tag{3}$$

Remark: The function $e(\tau)$ is called the einbein on the worldline of the particle.

a) How does e have to transform under reparameterisations $\tau \to \tilde{\tau}(\tau)$ in order to ensure the reparameterisation invariance of S?

b) Find the equation of motion for e. Insert the resulting equation into S and verify that S is (classically) equivalent to the action

$$S' = -mc \int_{\tau_i}^{\tau_f} d\tau \sqrt{-g(\dot{x}, \dot{x})}. \tag{4}$$

Exercise 3 – p-brane action

Show that the sigma-model form of the action for a p-brane,

$$S = -\frac{T_p}{2}\int d^{p+1}\sigma\, \sqrt{-h}\, h^{\alpha\beta}\, \eta_{\mu\nu}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu + \Lambda_p \int d^{p+1}\sigma\, \sqrt{-h} \; , \tag{5}$$

requires a non-vanishing cosmological constant Λ_p for $p \neq 1$. Here, X^μ are maps from the $(p+1)$-dimensional world volume of the p-brane into d-dimensional Minkowski spacetime ($\mu = 0, \ldots, d-1$).

Hint: Investigate whether the equation of motion for the world-volume metric is solved by equating the world-volume metric $h_{\alpha\beta}$ with the induced metric,

$$h_{\alpha\beta} = \eta_{\mu\nu}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu \quad , \quad \alpha, \beta = 1, \ldots, p+1 \; . \tag{6}$$