Instituto Superior Técnico/Departamento de Matemática/
Secção de Álgebra e Análise

Riemannian Geometry - 1st Semester 2010/11

2nd Exam - 01/02/2011

Duration: 3 hours

Answers in either Portuguese or English. Please justify all your answers.

(5 p) **1)** On $\mathbb{R}^2$ with the standard Cartesian coordinate system (x, y), consider the vector field

$$X = \begin{pmatrix} y + x\,(1 - x^2 - y^2) \\ -x + y\,(1 - x^2 - y^2) \end{pmatrix} .$$

(a) Show that for any $(x_0, y_0) \in \mathbb{R}^2 \backslash (0, 0)$ this vector field has a unique integral curve $\phi_t(x_0, y_0)$, defined for all $t \in \mathbb{R}$, such that $\phi_0(x_0, y_0) = (x_0, y_0)$.

Hint: Set $z = x + iy = re^{i\theta}$ and compute the integral curve in polar coordinates.

(b) Determine, according to the value $x_0^2 + y_0^2 > 0$, whether $\phi_t(x_0, y_0)$ (viewed as a map from $\mathbb{R}$ into $\mathbb{R}^2$) describes an immersion, an injective immersion or an embedding.

(5 p) **2)** Consider $\mathbb{R}^n$ with the standard Cartesian coordinate system $(x^1, \ldots, x^n)$ and with the standard metric $<,>$. Let S_{n-1} denote the set of points of $\mathbb{R}^n$ satisfying $\sum_{i=1}^n (x^i)^2 = 1$. Let $\Psi : S_{n-1} \to \mathbb{R}^n$ denote the inclusion map.

(a) On $\mathbb{R}^n - \{0\}$, consider the differential 1-form

$$\alpha = \left(\sum_{i=1}^n x^i\, dx^i \right) \left[\sum_{j=1}^n (x^j)^2 \right]^{-\frac{n}{2}} .$$

Compute $\star\alpha$, where $\star$ denotes the operator that associates to a p-form β an $(n-p)$-form $\star\beta$, as follows: $\star\,(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \varepsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n}\, dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n}$, where the antisymmetric symbol ε satisfies $\varepsilon_{12\ldots n} = 1$.

For $n = 3$, verify that $\star\alpha$ equals

$$\left(x^1\, dx^2 \wedge dx^3 - x^2\, dx^1 \wedge dx^3 + x^3\, dx^1 \wedge dx^2 \right) \left[\sum_{j=1}^3 (x^j)^2 \right]^{-\frac{3}{2}} .$$

(b) Show that $\star\alpha$ is closed, for any n.

(c) Set $n = 3$ and compute $\int_{S_2} \Psi^* \star \alpha$. Use spherical coordinates.

Is $\star\alpha$ exact (i.e. is $\star\alpha = d\gamma$ with γ a 1-form)?

(5 p) **3)** Let H be the hyperbolic plane $H = \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$ with Riemannian metric $g = y^{-2}\,(dx \otimes dx + dy \otimes dy)$. Let $z = x + iy$.

Show that if $f : H \to H$ is an orientation-preserving isometry, then it must be a holomorphic function of z.

Hint: Consider $f = u(x,y) + i\,v(x,y)$ with

$$df = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad , \quad a, b, c, d \neq 0 \ .$$

Using that f is an isometry of H, show that $ab + cd = 0$ as well as $a^2 + c^2 = b^2 + d^2$.

(5 p) **4)** Consider the following surface in $\mathbb{R}^3$,

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = h^2(z)\} \ ,$$

where $h(z) = 1 + \cosh z$. Let (r, θ, z) denote the usual cylindrical coordinates of $\mathbb{R}^3$.

(a) Show that the Riemannian metric on C induced by the Euclidean metric on $\mathbb{R}^3$ is

$$g = h^2(z)\, d\theta \otimes d\theta + \big(1 + h'(z)^2\big)\, dz \otimes dz \ .$$

(b) Show that its Gauss curvature K is given by

$$K = -\frac{1}{h(z)\,(h(z) - 1)^3} \ .$$

(c) Determine the vector obtained by parallel transporting $\frac{\partial}{\partial \theta}$ along the curve $\gamma(t) = (\cos t, \sin t, 1)$, $t \in [0, 2\pi]$.

(5 p) **5)** Let (M_n, g) be a compact, oriented Riemannian manifold of dimension $n > 1$ with Levi-Civita connection ∇. Let X be a C^∞ vector field on M. To X we associate the 1-form α such that $\alpha(Y) = g(X, Y)$ for all vectors $Y \in T(M_n)$. In local coordinates, $X = X^i\, \partial/\partial x^i$ and $\alpha = \alpha_i\, dx^i$.

(a) Consider a 1-form $w = w_i\, dx^i$. Show that $\mathcal{L}_X w = \left(X(w_j) + \frac{\partial X^i}{\partial x^j}\, w_i \right) dx^j$.

(b) Given an oriented Riemannian volume n-form η, show that $\mathcal{L}_X \eta = (\nabla_i X^i)\, \eta$.

(c) Show that a necessary and sufficient condition for $\mathcal{L}_X g = 0$ is $\nabla_i \alpha_j + \nabla_j \alpha_i = 0$.

Hint: define $\mathcal{L}_X g$ in analogy with (a).

$\Sigma = 25$ p

Formulae

- Cartan's structure equations for a field of orthonormal frames:

$$\begin{aligned} dw^i &= \sum_{j=1}^{n} w^j \wedge w_j{}^i \ , \\ w_i{}^j &= -w_j{}^i \ , \\ dw_i{}^j &= \Omega_i{}^j + \sum_{k=1}^{n} w_i{}^k \wedge w_k{}^j \ . \end{aligned}$$

- In two dimensions, $\Omega_i{}^j = -K\, w^i \wedge w^j$.
- On forms, $\mathcal{L}_X = i_X\, d + d\, i_X$.
- On functions, $\mathcal{L}_X f = X(f)$.
- Levi-Civita connection:

$$\Gamma^i{}_{jk} = \frac{1}{2} \sum_{l=1}^{n} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right) \quad , \quad \partial_l = \frac{\partial}{\partial x^l} \ .$$

- $\nabla_i X^k = \partial_i X^k + \Gamma^k{}_{ij}\, X^j$.
- $\nabla_i \alpha_j = \partial_i \alpha_j - \Gamma^k{}_{ij}\, \alpha_k$.
- Geodesic equation (in local coordinates $(x^1, \dots, x^n)$):

$$\ddot{x}^i + \sum_{j,k=1}^{n} \Gamma^i{}_{jk}\, \dot{x}^j\, \dot{x}^k = 0 \ .$$