Instituto Superior Técnico/Departamento de Matemática/
Secção de Álgebra e Análise

Riemannian Geometry - 1st Semester 2010/11

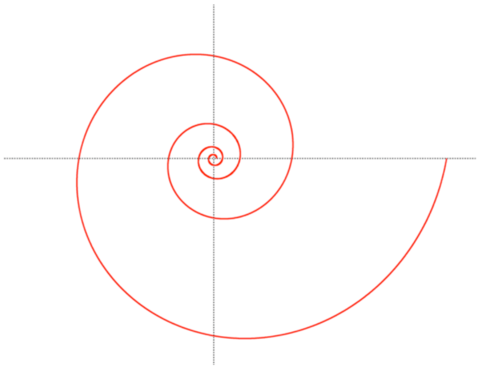
1st Exam - 19/01/2011

Duration: 3 hours

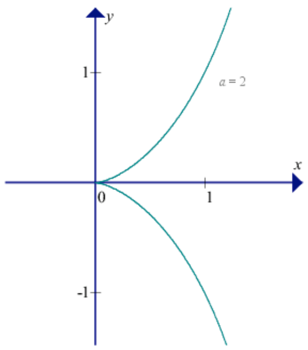
Answers in either Portuguese or English. Please justify all your answers.

(5 p) **1)** Consider the following maps $f : \mathbb{R} \to \mathbb{R}^2$,

1. $f(t) = (x(t), y(t)) = (\cos t, \sin t)$,
2. the logarithmic spiral $f(t) = (x(t), y(t)) = (e^t \cos t, e^t \sin t)$,



3. the cissoid $f(t) = (x(t), y(t)) = (\frac{2\,a\,t^2}{1+t^2}, \frac{2\,a\,t^3}{1+t^2})$ with $a = 2$,



Which of them describes an immersion, which an immersed submanifold and which an embedded submanifold?

(5 p) **2)** Consider $\mathbb{R}^3$ with the standard Cartesian coordinate system (x^1, x^2, x^3) and with the standard metric $<,>$. Let Ω be a bounded connected open set of $\mathbb{R}^3$ such that $\bar{\Omega}$ is a smooth manifold with boundary. Consider a smooth vector field on $\mathbb{R}^3$ given by

$$Y(x) = \sum_{i=1}^{3} Y^i(x) \frac{\partial}{\partial x^i} .$$

(a) Find a 2-form ω on $\mathbb{R}^3$ such that

$$d\omega = \sum_{i=1}^{3} \frac{\partial Y^i}{\partial x^i} dx^1 \wedge dx^2 \wedge dx^3 .$$

(b) Let $p \in \partial\bar{\Omega}$, and let (u, v) be local coordinates in a neighborhood $V \subset \partial\bar{\Omega}$ of p, such that $\{\frac{\partial}{\partial u}|_p, \frac{\partial}{\partial v}|_p\}$ is an orthonormal basis of $T_p\partial\bar{\Omega}$. Show that

$$\omega(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} Y^i U^j V^k ,$$

where $\{U^j\}$ and $\{V^k\}$ denote the components of $\frac{\partial}{\partial u}$ and of $\frac{\partial}{\partial v}$, respectively. The antisymmetric symbol ε_{ijk} is defined by $\varepsilon_{123} = 1$.

(c) Let n_p denote the normal unit vector at $p \in \partial\bar{\Omega}$ pointing outwards of Ω, and let $\{n_p, \frac{\partial}{\partial u}|_p, \frac{\partial}{\partial v}|_p\}$ denote an orthonormal basis of $T_p\mathbb{R}^3$ that is positively oriented. Show that

$$i^*\omega = < Y, n > du \wedge dv ,$$

where $i : \partial\bar{\Omega} \to \bar{\Omega}$ is the inclusion map.

Proceed as follows: (i) compute $i^*\omega$ to obtain an expression for n and (ii) subsequently check that n is a normal unit vector.

(d) Using the above, prove the well-known formula

$$\int_{\bar{\Omega}} \sum_{i=1}^{3} \frac{\partial Y^i}{\partial x^i} dV = \int_{\partial\bar{\Omega}} < Y, n > dA ,$$

where dV denotes the volume element of $\mathbb{R}^3$ and dA the area element of $\partial\bar{\Omega}$.

(5 p) **3)** Let M be a smooth manifold and let ∇ and $\tilde{\nabla}$ denote two linear connections (covariant derivatives) on M. We define

$$A(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y \quad , \quad B(X, Y) = A(X, Y) + A(Y, X) ,$$

where X, Y are smooth vector fields on M.

(a) Show that a necessary and sufficient condition for ∇ and $\tilde{\nabla}$ to have the same geodesics is that $B(X, Y) = 0$.

(b) Show that ∇ and $\tilde{\nabla}$ are identical if the two connections have the same geodesics and the same torsion.

(c) Now assume that ∇ and $\tilde{\nabla}$ have the same geodesics up to a change of parameterization of the geodesics. Show that $A(X, X)$ is proportional to X.

(5 p) **4)** Consider the Riemannian manifold (M, g, Γ) with $M = \mathbb{R}^2$, metric

$$g = \mathrm{e}^{2f(x,y)} \, (dx \otimes dx + dy \otimes dy) \ ,$$

and Levi-Civita connection Γ.

(a) Show that its Gauss curvature K is given by

$$K = -\mathrm{e}^{-2f(x,y)} \, (f_{xx} + f_{yy}) \quad , \quad f_{xx} = \frac{\partial^2 f}{\partial x \partial x} \ .$$

(b) Now take $\mathrm{e}^{-f} = \cosh x$, and show that the curve $c(t) = (x(t), y(t)) = (0, t)$ is a geodesic.

(5 p) **5)** Let G be a Lie group endowed with a Riemannian metric $<,>$ that is both left- and right-invariant, which is to say that left- and right multiplication are isometries for all $g \in G$, $L_g^* <,>=<,>$, $R_g^* <,>=<,>$. Let ∇ be the Levi-Civita connection. Let $i : G \to G$ be the diffeomorphism defined by $i(g) = g^{-1}$.

(a) Show that $(d\,i)_e = -\mathrm{id}$.

(b) Show that $i(g) = (R_{g^{-1}} \circ i \circ L_{g^{-1}})\,(g)$. Using this show that

$$(d\,i)_g = (dR_{g^{-1}})_e \, (d\,i)_e \, (d\,L_{g^{-1}})_g$$

for all $g \in G$.

(c) Conclude that i is an isometry, i.e.

$$< (d\,i)_g \, v, (d\,i)_g \, w >_{g^{-1}} = < v, w >_g$$

for all $v, w \in T_g G$.

(d) Consider two left-invariant vector fields X, Y. Using that all geodesics of G are the integral curves of left-invariant vector fields, show that

$$\nabla_X Y = \frac{1}{2} \, [X, Y] \ .$$

(e) Let X, Y, Z be left-invariant vector fields. Using (d), show that the Riemann tensor satisfies

$$R(X, Y) \, Z = \frac{1}{4} \, [Z, [X, Y]] \ .$$

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Formulae

- Cartan's structure equations for a field of orthonormal frames:

$$\begin{aligned} dw^i &= \sum_{j=1}^{n} w^j \wedge w_j{}^i \ , \\ w_i{}^j &= -w_j{}^i \ , \\ dw_i{}^j &= \Omega_i{}^j + \sum_{k=1}^{n} w_i{}^k \wedge w_k{}^j \ . \end{aligned}$$

- In two dimensions, $\Omega_i{}^j = -K\, w^i \wedge w^j$.
- Levi-Civita connection:

$$\Gamma^i{}_{jk} = \frac{1}{2} \sum_{l=1}^{n} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right) \quad , \quad \partial_l = \frac{\partial}{\partial x^l} \ .$$

- Geodesic equation (in local coordinates $(x^1, \ldots, x^n)$):

$$\ddot{x}^i + \sum_{j,k=1}^{n} \Gamma^i{}_{jk}\, \dot{x}^j\, \dot{x}^k = 0 \ .$$

- Torsion: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.
- Riemann tensor: $R(X,Y)\, Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.