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Operator Relations in Boundary Value Problems
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1. Operators associated with BVPs

A linear BVP in Eskin 73/81, Wloka 82, Hsiao-Wendland 08, e.g., is briefly written in the form

\[ Au = f \text{ in } \Omega \]  \hspace{1cm} \text{(PDE in nice domain)}

\[ Bu = g \text{ on } \Gamma = \partial\Omega \]  \hspace{1cm} \text{(boundary condition)}

**Associated operator**  \hspace{1cm} Boutet de Monvel 66, ... Wloka 82, ...

\[ L = \begin{pmatrix} A \\ B \end{pmatrix} : \mathcal{X} \rightarrow Y = Y_1 \times Y_2 \]  \hspace{1cm} \text{(data space)}

\[ \mathcal{X} = \text{(solution space)} \]

\( \mathcal{X} \) and \( Y \) are (at least) topological vector spaces. In this case the BVP is well-posed if and only if the associated operator is a linear homeomorphism (invertible and bi-continuous). Here we consider only Banach spaces.
So what is the (BV) Problem?

- To find a solution of $Lu = (f, g)$ (for just one set of given data)?
- To find all solutions (in $X$ for all data in $Y$)?
- To find solubility conditions and ...?
- To construct $L^{-1}$ (or a generalized inverse $L^{-}$) in closed form (or by series expansion, numerically, ...)?
- To prove well-posedness (or the Fredholm property etc ...)?
- To find suitable spaces for normal solubility etc ...?

All this has to do with Operator Relations!
2. Operator relations in BVPs - how they appear

We shall consider the following examples:

- Potential methods, idea of BIEs (what is ”equivalent reduction”)
- Reduction to semi-homogeneous systems (leading to equivalence after extension relations)
- Factorization of invertible operators in Banach spaces (abstract and classical WHOs in Sommerfeld problems)
- Normalization of singular operators (changing the spaces topology)
- Regularity of solutions (in scales of Sobolev spaces)
- Asymptotic behavior of solutions (from factor properties)
- Asymmetric factorization of scalar and matrix functions (for WH±HOs in diffraction from rectangular wedges)
- Structured matrix operators in wedge diffraction problems
Potential methods

The classical idea of potential theory (reduction to a simpler problem/operator) by a suitable potential ansatz

\[
L = \begin{pmatrix} A \\ B \end{pmatrix}
\]

\[H^1(\Omega) \xrightarrow{\mathcal{K}} Z \xrightarrow{T} Y\]

Operator composition \( T = L\mathcal{K} \) leads to the study of

- Boundary Integral Equations (BIE) \( \text{Hsiao and Wendland 08}\)
- Operator Factorization \( T = L\mathcal{K}, L = T\mathcal{K}^{-1} \) (simplest case),
- Operator Relations (for classes of ops.) in general \( \text{Castro 98}\).
Operator relations that appear frequently

**Equivalent operators** \( S \sim T \iff S = E T F \)

where \( E, F \) are boundedly invertible operators in Banach spaces.

**Equivalence after extension** \( \rightarrow \) [BGK 80] (\( \rightarrow \) minimal factorization)

\[
S \sim T \iff \begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} T & 0 \\ 0 & I_{Z_2} \end{pmatrix} F
\]

\( \Delta \) - related operators (appeared with WHHOs) \( \rightarrow \) [Castro 98]

\[
S \triangle T \iff \begin{pmatrix} S & 0 \\ 0 & S_{\Delta} \end{pmatrix} = E T F
\]

If \( E \) or \( F \) are only linear bijections (not necessarily bi-continuous), then \( S \) and \( T \) are called *algebraically equivalent*, etc, writing

\[
S \overset{\text{alg}}{\sim} T , \quad \overset{\text{alg}}{S^*} = T , \quad S \overset{\text{alg}}{\triangle} T
\]
3. Reduction to semihomogeneous systems

Consider the semihomogeneous (abstract) BVP

\[ L^0 u = \begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} 0 \\ g \end{pmatrix} \in \{0\} \times Y_2 \cong Y_2 \]

with associated operator

\[ B \mid_{\ker A} : \mathcal{X}_0 = \ker A \longrightarrow Y_2 \]

How is this operator related to the full thing

\[ L = \begin{pmatrix} A \\ B \end{pmatrix} : \mathcal{X} \longrightarrow Y = Y_1 \times Y_2 \]

In general, they will not be equivalent operators, since \( Y \) and \( Y_2 \) may not be isomorphic.

But, if \( A \) is surjective and \( \ker A \) is complemented, i.e., \( A : \mathcal{X} \longrightarrow Y_1 \)

is right invertible, then we have the following relation:
Reduction to semi-homogeneous systems
- considered as an operator relation

Lemma  Let  \( L = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{L}(\mathcal{X}, Y_1 \times Y_2) \) be a bounded linear operator in Banach spaces. Further let \( R \) be a right inverse of \( A \), i.e.,

\[
R \in \mathcal{L}(Y_1, \mathcal{X}) \quad , \quad AR = I.
\]

Then the following OF holds

\[
L = ETF = \begin{pmatrix} 0 & A|_{X_1} \\ I|_{Y_2} & B|_{X_1} \end{pmatrix} \begin{pmatrix} B|_{X_0} & 0 \\ 0 & I|_{X_1} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}
\]

where \( P = RA \), \( Q = I - RA \)  are continuous projectors in \( \mathcal{X} \), \( X_0 = \ker A = \ker P = \text{im } Q \), \( X_1 = \text{im } P = \ker Q \). The first and third factor are (boundedly) invertible as

\[
E = Y_2 \times X_1 \longrightarrow Y_1 \times Y_2
\]

\[
F = \mathcal{X} \longrightarrow X_0 \times X_1.
\]
Reduction to semi-homogeneous systems  
- an example for equivalence after extension

The previous Lemma is proved by verification. $A$ and $B$ are exchangeable. So we arrive at the following result, cf. S14:

**Theorem** Let $L = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{L}(\mathcal{X}, Y_1 \oplus Y_2)$ be a bounded linear operator in Banach spaces. Then

$$\exists R \in \mathcal{L}(Y_1, \mathcal{X}) \ A R = I \quad \Rightarrow \quad L \ \cong^* \ B \mid_{\ker A},$$

$$\exists R \in \mathcal{L}(Y_2, \mathcal{X}) \ B R = I \quad \Rightarrow \quad L \ \cong^* \ A \mid_{\ker B}. $$

There are several interpretations and conclusions, we mention a few.
Well-posedness and reduction to semi-homogeneous systems

**Corollary** The BVP is well-posed (i.e., $L$ is boundedly invertible) if and only if

1. the two semi-homogeneous problems are well-posed,
2. the solution splits uniquely as $u = u_0 + u^0$ where

\[
L_0 u_0 = \begin{pmatrix} f \\ 0 \end{pmatrix},
\]

\[
L^0 u^0 = \begin{pmatrix} 0 \\ g \end{pmatrix},
\]

3. $A$ and $B$ admit right inverses.

Each of the three conditions for its own is not sufficient for the BVP to be well-posed, but the first two or the last two conditions suffice.
What happens if $A$ or $B$ is not right invertible?

If $A$ is not right invertible, then
(i) $A$ is not surjective, the BVP is not solvable for all data $f \in Y_1$, i.e., $Y_1$ is chosen too large for a well-posed problem; or
(ii) $A$ is surjective but $\ker A$ not complemented, in which case it may help to change the topology of $Y_1$ or of $\mathcal{X}$.

Remarks

1. The right inverses $R$ of $A$ or $B$ in applications are often a volume or surface potential (see HW08) or an extension operator, left invertible to a trace operator, see Wloka 82 ... 

2. Each of the formulation has advantages in certain situations, see for instance Mikhailov: Boundary-domain integro-differential equations.
4. Equivalence after extension and matrical coupling
- a closer look at equivalence after extension relations

We identified the relation between \( L \) and \( B|_{\ker A} \) and \( A|_{\ker B} \) as \( \sim^* \) provided \( A \) and \( B \), respectively, are right invertible.

\[
S \sim^* T \iff \begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} T & 0 \\ 0 & I_{Z_2} \end{pmatrix} F
\]

**Remark** This kind of operator relation (which belongs to the class of so-called *operator matrix identities*) is very important in theory and applications. It appears frequently in "substitution, factorization, extension and reduction methods".

We study some of their properties. Of course, it is an equivalence relation (in the genuine sense), i.e., reflexive, symmetric and transitive. However there is much more.
Theorem of Bart and Tsekanovskii

**Theorem BT - part 1** Let $T \in \mathcal{L}(X_1, X_2)$ and $S \in \mathcal{L}(Y_1, Y_2)$ be bounded linear operators in Banach spaces and assume $T \sim S$.

Then $\ker T \cong \ker S$. Also $\text{im } T$ is closed if and only if $\text{im } S$ is closed, and in that case $X_2/\text{im } T \cong Y_2/\text{im } S$.

**Remark** This conclusion is most important in applications and often simplifies the reasoning tremendously.
Transfer properties of an EAE relation

**Transfer property 1** If \( S \sim^* T \), the two operators belong to the same regularity class: They are only simultaneously invertible, Fredholm, generalized invertible, normally solvable, left invertible, etc.

**Transfer property 2** If the inverses of the operators \( E, F \) in the relation \( S \sim^* T \) are known, a generalized inverse of \( T \) can be computed from a generalized inverse of \( S \), namely by a ”reverse order law”

\[
\begin{pmatrix}
S & 0 \\
0 & I_{Z_1}
\end{pmatrix} = E \begin{pmatrix}
T & 0 \\
0 & I_{Z_2}
\end{pmatrix} F \Rightarrow T^{-} = R_{11} F^{-1} \begin{pmatrix}
S^{-} & 0 \\
0 & I_{Z_1}
\end{pmatrix} E^{-1}
\]

where \( R_{11} \) denotes the restriction to the first block of the operator matrix.

More results can be found in the PhD thesis of Castro 1998 et al., such as normalization, asymptotic expansion, ...
The Theorem of Bart and Tsekanovskii:
- part 2, inverse conclusion

**Theorem BT - part 2** Let $T \in \mathcal{L}(X_1, X_2)$ and $S \in \mathcal{L}(Y_1, Y_2)$ be bounded linear operators in Banach spaces and assume that $T$ and $S$ are generalized invertible*.

Then $T \sim S$ if and only if $\ker T \cong \ker S$ and $X_2 / \im T \cong Y_2 / \im S$.

*This assumption is essential (there is an example where sufficiency fails otherwise, see BT 1992).
Matrical coupling   Bart, Gohberg, Kaashoek 1984

**Theorem**   BGK 84   Let $S \in \mathcal{L}(X_1, Y_1)$, $T \in \mathcal{L}(X_2, Y_2)$. Then $S \sim^* T$ if the two operators are *matrically coupled*, i.e.,

$\begin{pmatrix} S & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \end{pmatrix}^{-1} \begin{pmatrix} * & T \end{pmatrix}.$

In symmetric setting, this is just an interpretation of the formula $PAP + Q \sim P + QA^{-1}Q$, see BGK 85, which is well known.

**Theorem**   Bart-Tsekanovsny 1992

$S \sim^* T$ iff $S$ and $T$ are matrically coupled.

**Remark**   As we know already: This implies that

$$\ker S \cong \ker T, \quad \coker S \cong \coker T$$

and the inverse conclusion holds if both operators $S$ and $T$ are generalized invertible.
Examples of BVPs which are matrically coupled more precisely: whose associated operators are matrically coupled

Diffraction of time-harmonic waves from plane screens in $\mathbb{R}^3$.

Given a 2D special Lipschitz domain $\Sigma$ (see Stein 70) and $g \in H^{1/2}(\Sigma)$ (the trace of a primary field) we look for the weak solution of the Dirichlet problem

$$\Delta + k^2 u = 0 \quad \text{in} \quad \Omega = \mathbb{R}^3 \setminus \Gamma \quad \text{where} \quad \Gamma = \overline{\Sigma} \times \{0\}$$

$$T_0 \ u = g \quad \text{on} \quad \Gamma = \partial \Omega \ (\text{both banks})$$

The Neumann problem for the complementary screen $\Sigma_* = \mathbb{R}^2 \setminus \overline{\Sigma}$ is briefly written as

$$\Delta + k^2 u = 0 \quad \text{in} \quad \Omega_* = \mathbb{R}^3 \setminus \Gamma_* \quad \text{where} \quad \Gamma_* = \overline{\Sigma_*} \times \{0\}$$

$$T_1 u = \partial u / \partial x_3 \big|_{x_3=0} = h \quad \text{on} \quad \overline{\Sigma_*} = \partial \Omega_* \ (\text{both banks})$$

where $h \in H^{-1/2}(\Sigma_*)$ is given.
An abstract Babinet principle

Theorem The operators associated to the last two BVPs are matrically coupled and (therefore) equivalent after extension:

\[ L_{D,\Omega} \sim^* L_{N,\Omega_*}. \]

The proof is based upon what follows:
Relations between ”General Wiener-Hopf operators”

Details can be seen in Appendix III.
5. Factorization methods for singular operators related to boundary integral equations, ...

- Rich history: Muskhelishvili, Gakhov, Vekua, Mikhlin, Gohberg-Krein, Simonenko, Prössdorf, Coburn, Douglas, Devinatz-Shinbrot, Spitkovsky, ...
- Many operator classes: Wiener-Hopf, Toeplitz, Riemann problems, abstract settings, ...
- Plenty of constructive methods: Scalar and matrix functions, rational functions, decomposing algebras, generalized factorization, ... LU, polar decomposition, ...
- Wide field of applications in Math. Physics: Acoustics, Aerodynamics, Elasticity theory, ...
General Wiener-Hopf operators (WHO$\text{s}$)

Here we consider some algebraic methods, which become constructive when combined with explicit factorization of matrix functions. The following type of operators are so-called general WHO$\text{s}$ (Shinbrot 1964), abstract Wiener-Hopf operators (Cebotarev 1967), or projections of operators (Gohberg-Krupnik 1973/79)

\[ W = T_P(A) = PA|_{PX} : PX \to PX (= \text{im} \, P) \]

where $X$ is a Banach (or even Hilbert) space, $P = P^2 \in \mathcal{L}(X)$ a bounded projector, $A \in \mathcal{GL}(X)$ an invertible linear operator.

Variants which partly admit the same results:

\[ W = P_2A|_{P_1X} : P_1X \to P_2Y \text{ asymmetric WHO} \]

where $X, Y$ are a Banach spaces, $P_1, P_2$ bounded projector in $X$ and $Y$, respectively and $A \in \mathcal{L}(X,Y)$ an invertible linear operator; further

\[ w = pap \]

where $a, p \in \mathcal{R}$, which is a unital algebra, $a$ invertible and $p^2 = p$. 
The cross factorization theorem

**Theorem S 83** Let \( W = T_P(A) = PA|_{PX} \) be a general WHO (i.e.) where \( X \) is a Banach space, \( A \in \mathcal{GL}(X) \), \( P^2 = P \in \mathcal{L}(X) \). Then \( W \) is generalized invertible if and only if

\[
A = A_- C A_+
\]

where \( A_\pm \in \mathcal{GL}(X) \), \( A_+ PX = PX \), \( A_- QX = QX \) \( (Q = I - P) \) and \( C \) splits the space \( X \) twice into four subspaces such that

\[
X = \begin{cases} PX \\ X_1 & + & X_0 \\ \end{cases} + \begin{cases} QX \\ X_2 & + & X_3 \\ \end{cases}
\]

\[
= Y_1 + \begin{cases} PX \\ Y_2 \end{cases} + \begin{cases} QX \\ Y_0 & + & Y_3 \end{cases}
\]

where \( C \) maps each \( X_j \) onto \( Y_j \), \( j = 0, 1, 2, 3 \), i.e., \( X_0 = C^{-1}QCPX \), \( X_1 = C^{-1}PCPX \) etc. In this case,

\[
W^- = PA_+^{-1}PC^{-1}PA_-^{-1}|_{PX} \text{ is a GI of } W.
\]
The cross factorization theorem - sketch of proof

The sufficient part is quickly done by verification in a few lines. The necessary part works with space decomposition and reduction to a one-sided invertible, restricted operator, it needs one page and a half. Both are done for the asymmetric version, see S 83.

The sufficient part for the algebra setting is as simple as before. Necessity of the factorization consists in guessing a cross factorization from a generalized inverse, i.e. if \( wvw = w \), then

\[
\begin{align*}
a &= a_- c a_+ \\
&= [e + qav] [a - ava + w + a(p - vw)a^{-1}(p - vw)a] \times \\
&\times [e + vaq - (p - vw)a^{-1}(p - vw)a]
\end{align*}
\]

represents a cross factorization of \( a \).

However verification needs almost two pages, see S 85.
Historic remarks on general WHOs

Shinbrot 1964 : Definition and basic properties of general WHOs.

Cebotarev 1967 : One-sided invertibility in the algebraic setting.

Devinatz and Shinbrot 1969 : Invertibility of (symmetric) general WHOs in separable Hilbert spaces.


Speck 1983 : Symmetric version of the cross factorization theorem.

Speck 1985 : Asymmetric and algebraic versions.

Ferreira dos Santos 1988 : A geometric perspective ... 

**Associated WHOs** \( QA^{-1}|_{QX} \)

were discussed for Hilbert space operators by Devinatz-Shinbrot 1969

The following observation was made by Speck 1984 during a conference in Oberwolfach, Germany, after the talk of I. Gohberg.

**Remark** *Associated general WHOs* are equivalent after extension:

\[
P A|_{_P X} \sim Q A^{-1}|_{Q X}
\]

where \( X \) is a Banach space, \( A \in \mathcal{GL}(X) \), \( P^2 = P = I - Q \in \mathcal{L}(X) \). This well-known relation has a certain similarity with matricial coupling (in symmetric space settings \( X_j = Y_j \)).

**Proof**

\[
P A P + Q = (I - P A Q)(P A + Q) = (I - P A Q)(P + Q A^{-1}) A
\]

\[
= (I - P A Q)(I + Q A^{-1} P)(P + Q A^{-1} Q) A.
\]

The remark was published and commented by BGK 1985 in an addendum in IEOT 8 (1985) 890-891.
Variants of the Cross Factorization Theorem

**Theorem** (WH factorization through an intermediate space) S 14

Let $ W = P_2 A|_{P_1 X} $ be an *asymmetric WHO* (i.e.) where $ X, Y $ are Banach spaces, $ A \in \mathcal{L}(X, Y) $ invertible, $ P_2^1 = P_1 \in \mathcal{L}(X), P_2^2 = P_2 \in \mathcal{L}(Y) $. Then $ W $ is generalized invertible if and only if there exists a Banach (intermediate) space $ Z $ and a projector $ P \in \mathcal{L}(X) $ such that $ A $ splits into invertible operators

$$ A = A_- C A_+ $$

$$ Y \leftarrow Z \leftarrow Z \leftarrow X $$

where $ A_+ $ maps $ P_1 X $ onto $ PZ $, $ A_- $ maps $ QZ $ onto $ Q_2 Y $, and $ C $ is a cross factor (in modified obvious setting). In this case,

$$ W^- = P_1 A_+^{-1} PA^-1 \big|_{P_2 Y} \text{ is a GI of } W. $$

**Remark** This includes operators acting between Sobolev spaces of different order and Simonenko’s concept of generalized factorization.
6. Wiener-Hopf operators in Sobolev spaces
   - originated from diffraction theory

\[
W = r_+ A : L^2_+ \rightarrow L^2(\mathbb{R}_+)
\]

\[
A = \mathcal{F}^{-1} \Phi_A \cdot \mathcal{F}, \quad \Phi_A \in C^\mu(\mathbb{R}) \cap \mathcal{G}L^\infty(\mathbb{R})
\]

\[
W_s = \begin{cases} 
\text{Rst } W : H^s_+ \rightarrow H^s(\mathbb{R}_+), \ s > 0 \\
\text{Ext } W : H^s_+ \rightarrow H^s(\mathbb{R}_+), \ s < 0 
\end{cases}
\]

Then it is well known that (see Eskin 1973, Duduchava 1979, ...)

- \( W \sim \ell_0 r_+ A : L^2_+ \rightarrow L^2_+ \) which has the form of a general WHO,
- a cross factorization \( A = A_- CA_+ \) exists explicitly provided \( \Phi_A(\infty)/\Phi_A(-\infty) \notin \mathbb{R}_- \), i.e., \( \Phi_A \) is 2-regular,
- a generalized inverse \( W_s^- = ... \) is explicitly obtained provided the lifted symbol \( (\xi - i)^s \Phi_A(\xi)(\xi + i)^{-s} \), is 2-regular.
Explicit formulas for $W^-$

Instead of the Fourier symbol $\Phi$ consider the index-free function (see Duduchava 1979)

$$
\Phi_0 = \zeta^{-\omega} \Phi^{-1}(+\infty) \Phi
$$

where

$$
\omega = \frac{1}{2\pi i} \int_{\mathbb{R}} d \log \Phi , \quad \Re \omega + \frac{1}{2} \notin \mathbb{Z} , \quad \zeta(\xi) = \frac{\lambda_-(\xi)}{\lambda_+(\xi)} = \frac{\xi - i}{\xi + i}.
$$

$\Phi_0$ admits a canonical Wiener-Hopf factorization in $C^\mu(\dot{\mathbb{R}})$

$$
\Phi_0 = \Phi_{0-} \Phi_{0+} , \quad \Phi_{0\pm} = \exp\{P_\pm \log \Phi_0\}
$$

where $P_\pm$ denote the Hilbert projections $P_\pm = (I \pm S_{\mathbb{R}})/2$. A generalized factorization (see Simonenko 1968) is given by

$$
\Phi = \Phi_- \cdot \zeta^{\kappa} \cdot \Phi_+ = \lambda_-^{-\kappa} \Phi_{0-} \cdot \zeta^{\kappa} \cdot \lambda_+^{-\omega + \kappa} \Phi_{0+} \Phi(+\infty)
$$

$$
\kappa = \max\{z \in \mathbb{Z} : z \leq \Re \omega + \frac{1}{2}\}.
$$
Now a cross factorization of $A$ is obtained putting

$$A = A_- C A_+ = \mathcal{F}^{-1} \Phi_- \cdot \mathcal{F} \quad \mathcal{F}^{-1} \zeta^\kappa \cdot \mathcal{F} \quad \mathcal{F}^{-1} \Phi_+ \cdot \mathcal{F}.$$ 

It represents a bounded operator factorization through the Sobolev space $Z = H^{Re\omega - \kappa}$.

A generalized inverse of $W$ is given by

$$W^- = A_+^{-1} P_Z C^{-1} P_Z A_-^{-1} \ell_0.$$ 

where $P_Z$ is the extension/restriction of the Hilbert projection on $Z$.

Further it enables asymptotic results in terms of an expansion of the generalized inverse in a scale of Sobolev spaces, using the chain of generalized inverses of $W_s$, see Penzel-S 1993.
7. Factor properties, intermediate spaces and singularities

There is a close relation between properties of the factors, here particularly their increase at infinity, the kind of intermediate space, and the asymptotic behavior of solutions at zero.

Example: Classical, scalar WHO with analytical index zero

\[ W = r_+ A : L^2_+ \rightarrow L^2(\mathbb{R}_+) \]

\[ A = \mathcal{F}^{-1} \Phi_A \cdot \mathcal{F} , \quad \Phi_A \in C^\mu(\mathbb{R}) \cap \mathcal{G}L^\infty(\mathbb{R}) \]

\[ A = A_- A_+ , \quad \Phi_A = \Phi_- \Phi_+ \quad \text{(generalized fact.)} \]

\[ Y \leftarrow Z \leftarrow X , \quad Z = H^{\text{Re} \omega} , \quad \text{Re} \omega \in ]-1/2, 1/2[. \]

For \( g \in C^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \), the singular behavior of the solution is

\[ W^{-1} g(x) = A_+^{-1} P A_-^{-1} \ell_0 g(x) \sim |x|^{-\text{Re} \omega} \quad \text{as} \quad |x| \rightarrow 0. \]

In the system’s case there appear log terms, as well, see Castro 95.
Minimal normalization of WHOs

**Theorem** MoST 1998

Let $W$ as before, $s \in \mathbb{R}$ be critical, i.e., $W_s$ not Fredholm. Then $W_{s+\varepsilon}$ is Fredholm for $\varepsilon \in ]0, 1/2[$ with generalized inverse $W_{s+\varepsilon}^-$ given by factorization (usual formula) and $W_s$ can be *image normalized* replacing the image space $H^s(\mathbb{R}_+)$ by a proper dense subspace $\tilde{H}^s(\mathbb{R}_+) = r_+ \Lambda_{-s}^{-1/2} H_+^{-1/2} \subset H^s(\mathbb{R}_+)$ such that the restricted operator

$$\tilde{W}^s = \text{Rst} W_s : H_+^s \to \tilde{H}^s(\mathbb{R}_+)$$

is Fredholm and has a GI given by

$$(\tilde{W}^s)^- = \text{Ext} W_{s+\varepsilon}^- : \tilde{H}^s(\mathbb{R}_+) \to H_+^s.$$

**Remark** There is an analog for *domain normalization*, matrix cases, transfer of normalization by operator relations like $\sim^*$ etc.
8. **Convolution type operators with symmetry (CTOS)**

alias Wiener-Hopf±Hankel operators

\[ T = r_+ A(I \pm J) \ell_0 = r_+ A \ell^c : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \]

\[ A = \mathcal{F}^{-1} \Phi_A \cdot \mathcal{F}, \quad \Phi_A \in C^{\mu}(\mathbb{R}) \cap GL^{\infty}(\mathbb{R}) \]

\[ Jf(x) = f(-x), \quad x \in \mathbb{R}, \quad \ell^c = \ell^e \text{ or } \ell^o \text{ (even/odd extension)} \]

Then

- \( T \sim \ell_0 r_+ A : L^2_{e/o} \to L^2_+ \) which is understood as general WHO,
- a cross factorization \( A = A_- CA_e \) exists explicitly provided \( \Phi_A(+\infty)/\Phi_A(-\infty) \not\in e^{i\pi/2}\mathbb{R}_+ \) resp. \( \not\in e^{3i\pi/2}\mathbb{R}_+ \),
- and then a generalized inverse \( W_s^- = ... \) is explicitly obtained by asymmetric factorization, see CST 2004.

Asymmetric factorization (scalar case): 
a direct approach to the inversion of CTOS

The idea comes from Basor and Ehrhardt 2004. Instead of the Fourier symbol $\Phi$ consider the index-free function (see Duduchava 1979)

$$\Psi = \zeta^{-\omega} \Phi^{-1}(+\infty) \Phi$$

where

$$\omega = \frac{1}{2\pi i} \int_{\mathbb{R}} d \log \Phi , \quad \Re(\omega) \pm \frac{1}{4} \notin \mathbb{Z} , \quad \zeta(\xi) = \frac{\lambda_-(\xi)}{\lambda_+(\xi)} = \frac{\xi - i}{\xi + i} .$$

Putting $\tilde{\Psi} = J\Psi$ define the symmetrized function

$$G = \Psi \tilde{\Psi}^{-1} \in C^\mu(\mathbb{R}) \quad \text{with} \quad \text{ind} \ G = 0,$$

which admits an antisymmetric Wiener-Hopf factorization in $C^\mu(\mathbb{R})$

$$G = G_- \ G_+ = G_- \ G_-^{-1} .$$
From antisymmetric to asymmetric factorization (scalar)

Now a cross factorization is obtained putting (see CST04)

\[
\Phi = \Phi_- \zeta^\kappa \Phi_e, \quad \kappa = \max\{z \in \mathbb{Z} : z \leq \Re e(\omega) \pm \frac{1}{4}\}
\]

\[
\Phi_- = \lambda_2^{2(\omega-\kappa)} \exp\{P_- \log G\}, \quad \Phi_e = \zeta^{-\kappa} \Phi_-^{-1} \Phi.
\]

It can be proved that \(\Phi_e\) is an even function!

The above sign of \(\pm\) depends on which version we consider:

\[
T = r_+ A(I \pm J)\ell_0 = r_+ A\ell^c : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)
\]

\[
\ell^c = \ell^e \text{ or } \ell^o \text{ (even/odd extension)}
\]
9. Structured matrix operators (by example):

ΨDOs occurring in BVPs for the HE in a quarter-plane

\begin{align*}
Q_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \ x_2 > 0\} \\
\Gamma_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \ x_2 = 0\} \\
\Gamma_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, \ x_2 > 0\}
\end{align*}

Determine (all weak solutions) \( u \in H^1(Q_1) \) (explicitly and in closed analytical form) such that

\[
Au(x) = (\Delta + k^2)u(x) = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k^2 \right) u(x) = 0 \quad \text{in} \quad Q_1
\]

\[
B_1 u(x) = \left( \alpha u + \beta \frac{\partial u}{\partial x_2} + \gamma \frac{\partial u}{\partial x_1} \right)(x) = g_1(x) \quad \text{on} \quad \Gamma_1
\]

\[
B_2 u(x) = \left( \alpha' u + \beta' \frac{\partial u}{\partial x_1} + \gamma' \frac{\partial u}{\partial x_2} \right)(x) = g_2(x) \quad \text{on} \quad \Gamma_2.
\]

Sometimes it is useful to consider "small regularity": \( u \in H^{1+\varepsilon}(Q_1) \).
BVPs for the HE in a quarter-plane ctd

Herein the following data are given: a complex wave number $k$ with $\Re m k > 0$, constant coefficients $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ as fixed parameters and arbitrary $g_j \in H^{-1/2}(\Gamma_j)$. Note that $\beta$ and $\beta'$ are the coefficients of the normal derivatives, whilst $\gamma$ and $\gamma'$ are those of the tangential derivatives. In case of a Dirichlet condition, i.e., $\beta = \gamma = 0$, we assume $g_1 \in H^{1/2}(\Gamma_1)$.

The space of weak solutions of the HE is denoted by

$$\mathcal{H}(Q_1) = \{ u \in H^1(Q_1) : (\Delta + k^2)u = 0 \} = \ker A$$

in the previous notation and we shall consider the operator $L^0$ associated to the semi-homogeneous problem.
The impedance problem

The *impedance problem* shows the following boundary conditions:

\[
\mathcal{S}_1 u(x) = \frac{\partial u(x)}{\partial x_2} + ip_1 u(x) = g_1(x) \quad \text{on} \quad \Gamma_1
\]

\[
\mathcal{S}_2 u(x) = \frac{\partial u(x)}{\partial x_1} + ip_2 u(x) = g_2(x) \quad \text{on} \quad \Gamma_2
\]

where the imaginary part of \( p_j \) turns out to be important:

1. \( \Im p_j > 0 \): physically most reasonable due to positive finite conductance in electromagnetic theory for instance;

2. \( p_1 = 0 \) or/and \( p_2 = 0 \): *Neumann condition(s)* allow a much simpler solution, MPST 93, CST 04;

3. if both \( \Im p_j \) are negative the potential approach has to be modified in a cumbersome way (in contrast to the mixed case which can be solved like (1));

4. if \( p_j \in \mathbb{R} \setminus \{0\} \) for \( j = 1, 2 \), the problem needs another kind of normalization (LAP) that is not carried out here.
Compatibility conditions

The Dirichlet problem (DD) is only solvable under certain compatibility conditions (see Hsiao-Wendland 2008) for the Dirichlet data

\[ g_1 - g_2 \in \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+), \]

i.e., this function is extendible by zero onto the full line \( \mathbb{R} \) such that the zero extension \( \ell_0(g_1 - g_2) \) belongs to \( H^{1/2+\varepsilon}(\mathbb{R}) \). The Neumann problem (NN) needs a compatibility condition

\[ g_1 + g_2 \in \tilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+), \]

if and only if \( \varepsilon = 0 \). The mixed problems (DN) do not require any additional condition.
Solution of the DN problem and half-line potentials (HLPs)

The solution is amazingly simple in the DN case:

\[ u(x_1, x_2) = \mathcal{F}_{\xi \mapsto x_1}^{-1} e^{-t(\xi) x_2} \ell^e g_1(\xi) - \mathcal{F}_{\xi \mapsto x_2}^{-1} e^{-t(\xi) x_1} t^{-1}(\xi) \ell^o g_2(\xi) \]

where \( \ell^e \) and \( \ell^o \) denote even and odd extension.

**Theorem**  MPST 93, CST 04

The following mapping

\[ \mathcal{K}_{DN,Q_1} : X = H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_2) \to \mathcal{H}^1(Q_1) \]

\[ u = \mathcal{K}_{DN,Q_1}(f, g)^T = \mathcal{K}_{D,Q_{12}} \ell^e f + \mathcal{K}_{N,Q_{14}} \ell^o g \]

\[ \mathcal{K}_{D,Q_{12}} \ell^e f(x) = \mathcal{F}_{\xi \mapsto x_1}^{-1} \exp [-t(\xi) x_2] \ell^e f(\xi), \quad x \in Q_{12} \]

\[ \mathcal{K}_{N,Q_{14}} \ell^o g(x) = -\mathcal{F}_{\xi \mapsto x_2}^{-1} \exp [-t(\xi) x_1] t^{-1}(\xi) \ell^o g(\xi), \quad x \in Q_{14} \]

is a toplinear isomorphism that satisfies

\[ (T_0,\Gamma_1, T_1,\Gamma_2)^T \mathcal{K}_{DN,Q_1} = I_X \]

\[ \mathcal{K}_{DN,Q_1}(T_0,\Gamma_1, T_1,\Gamma_2)^T = I_{\mathcal{H}^1(Q_1)}. \]
Resolvent of the DN problem as potential operator

- Using this representation as a potential operator, it was possible to solve explicitly a great number of BVPs, see CST 2004.
- The reason was that the corresponding boundary \( \Psi \)DO a matricial \( 2 \times 2 \) structured operator that has a triangular form in many cases.
- It also gave the idea to introduce so-called *half-line potentials* (HLPs):
Half-line potentials (HLPs)  

Let $m_j \in \mathbb{N}_0$, $\psi_j : \mathbb{R} \to \mathbb{C}$ be measurable functions such that $\psi_j$ is $m_j$-regular, i.e., $t^{-m_j} \psi_j \in \mathcal{G}L^\infty$ and let $\ell_j : H^{1/2-m_j}(\mathbb{R}_+) \to H^{1/2-m_j}(\mathbb{R})$ be continuous extension operators for $j = 1, 2$. Then

$$u(x) = \mathcal{F}_{\xi \mapsto x_1}^{-1} \left\{ \exp[-t(\xi)x_2] \psi_1^{-1}(\xi) \widehat{\ell_1f_1(\xi)} \right\} + \mathcal{F}_{\xi \mapsto x_2}^{-1} \left\{ \exp[-t(\xi)x_1] \psi_2^{-1}(\xi) \widehat{\ell_2f_2(\xi)} \right\}$$

with $f_j \in H^{1/2-m_j}(\mathbb{R}_+)$ and $x = (x_1, x_2) \in Q_1$ is said to be a half-line potential (HLP) in $Q_1$ with density $(f_1, f_2)$.

We call it strict for $\mathcal{H}^1(Q_1)$ if it defines a bijective mapping, writing

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 = \mathcal{K}^{\psi_1, \psi_2}$$

$$: \quad X = H^{1/2-m_1}(\mathbb{R}_+) \times H^{1/2-m_2}(\mathbb{R}_+) \to \mathcal{H}^1(Q_1)$$

specifying $\ell_j$ when necessary. Keeping in mind low regularity properties: $\mathcal{K}^\varepsilon : X^\varepsilon = H^{1/2-m_1+\varepsilon}(\mathbb{R}_+) \times H^{1/2-m_2+\varepsilon}(\mathbb{R}_+) \to \mathcal{H}^{1+\varepsilon}(Q_1)$, we speak about a strict HLP for $\mathcal{H}^{1+\varepsilon}(Q_1)$ in the corresponding case.
Half-line potentials (HLPs)  

Proposition  

Let $L = (B_1, B_2)^T$ and $\mathcal{K}$ be given as before. Then the composed operator $T = L\mathcal{K}$ has the form:

$$T = \begin{pmatrix} r + A\phi_{11} \ell_1 & C_0 A\phi_{12} \ell_2 \\ C_0 A\phi_{21} \ell_1 & r + A\phi_{22} \ell_2 \end{pmatrix} : X \to Y$$

where $Y = H^{-1/2}(\mathbb{R}_+)^2$ identifying $\Gamma_j$ with $\mathbb{R}_+$ and

$$\phi_{11} = \sigma_1 \psi_1^{-1} = (\alpha - \beta t + \gamma \vartheta) \psi_1^{-1}, \quad \phi_{12} = \sigma_1 \psi_2^{-1} = (\alpha + \beta \vartheta - \gamma t) \psi_2^{-1}$$

$$\phi_{21} = \sigma_2 \psi_1^{-1} = (\alpha' + \beta' \vartheta - \gamma' t) \psi_1^{-1}, \quad \phi_{22} = \sigma_2 \psi_2^{-1} = (\alpha' - \beta' t + \gamma' \vartheta) \psi_2^{-1}$$

The main diagonal contains CTOS if $\ell_j = \ell^{e/o}$. The others are Fourier integral operators (combined with extensions) defined for any $\phi \in L^\infty$ by

$$K^{(s)} = C_0 A\phi \ell^o : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+) \quad , \quad s \in ] - 3/2, 1/2 [, \quad K^{(s)} f(x_1) = (2\pi)^{-1} \int_\mathbb{R} \exp[-t(\xi)x_1] \phi(\xi) \widehat{\ell^o f(\xi)} \, d\xi , \quad x_1 \in \mathbb{R}_+.$$
They are well-defined and bounded if $s \in ] - 3/2, 1/2[$. In this case, $K^{(s)} = 0$ if and only if $\phi$ is an even function. Replacing $\ell^o$ by $\ell^e$, we have boundedness of $K^{(s)}$ for $s \in ] - 1/2, 3/2[$ being zero if and only if $\phi$ is odd.

\[
L = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}
\]

\[
\mathcal{H}^1(Q_1) \xrightarrow{L} Y
\]

\[
\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2
\]

\[
X \xleftarrow{T} T = \begin{pmatrix} T_1 & K_1 \\ K_2 & T_2 \end{pmatrix}
\]
Problems solvable by the DN ansatz

**Theorem CST 04** The following classes of interior wedge problems can be explicitly solved by generalized inversion of $T$ provided $T$ is of normal type:

<table>
<thead>
<tr>
<th>$\Gamma_1 \setminus \Gamma_2$</th>
<th>$D$</th>
<th>$N$</th>
<th>$I$</th>
<th>$\mathcal{T}$</th>
<th>$O$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>$\Pi^+$</td>
<td>$O$</td>
<td>$I$</td>
<td>$\Pi^*_\pm$</td>
<td>$\text{III}$</td>
<td>$\text{III}$</td>
</tr>
<tr>
<td>$N$</td>
<td>$O^*$</td>
<td>$\Pi^+$</td>
<td>$\Pi^*$</td>
<td>$I^*$</td>
<td>$\text{III}^*$</td>
<td>$\text{III}^*$</td>
</tr>
<tr>
<td>$I$</td>
<td>$I^*$</td>
<td>$\Pi^+$</td>
<td>$\Pi^*$</td>
<td>$I^*$</td>
<td>$\text{III}^*$</td>
<td>$\text{III}^*$</td>
</tr>
<tr>
<td>$\mathcal{T}$</td>
<td>$\Pi^*_\pm$</td>
<td>$I$</td>
<td>$I^*$</td>
<td>$\text{IV}$</td>
<td>$\text{IV}$</td>
<td>$\text{IV}$</td>
</tr>
<tr>
<td>$O$</td>
<td>$\text{III}^*$</td>
<td>$\text{III}$</td>
<td>$\text{IV}^*$</td>
<td>$\text{IV}$</td>
<td>$\text{IV}^*$</td>
<td></td>
</tr>
<tr>
<td>$G$</td>
<td>$\text{III}^*$</td>
<td>$\text{III}$</td>
<td>$\text{IV}^*$</td>
<td>$\text{IV}^*$</td>
<td>$\text{IV}^*$</td>
<td>$\text{IV}^*$</td>
</tr>
</tbody>
</table>
Legend : (referring to CST 04)

**bold** – reference problems discussed in detail.

* – belongs to the corresponding class where the two variables are exchanged, see (2.11).

O – invertible by representation formulas (2.11).

I – direct inversion by Theorem 3.2 (even pre-symbol), Example 3.3.

I− – dito (minus type), Example 3.5.

I+ – right inversion via AFIS, see Proposition 5.3, cf. Corollary 5.6.

II, II± – like I, I± after image normalization, Example 3.4.

III – Fredholm and one-sided invertible via AFIS, eventually after normalization, see Theorem 5.4, Theorem 5.5, only one scalar factorization needed.

IV – similar with two scalar operators, see Corollary 5.6

empty spaces – correspond with open problems, not decomposing (triangular) in the preceding sense.
In case of the remainder BVPs in the above diagram, one needs a different ansatz in order to end up with a triangular matrix operator. Namely, given (as before)

\[ Au(x) = (\Delta + k^2)u(x) = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k^2 \right) u(x) = 0 \quad \text{in} \quad Q_1 \]

\[ B_1 u(x) = \left( \alpha u + \beta \frac{\partial u}{\partial x_2} + \gamma \frac{\partial u}{\partial x_1} \right)(x) = g_1(x) \quad \text{on} \quad \Gamma_1 \]

\[ B_2 u(x) = \left( \alpha' u + \beta' \frac{\partial u}{\partial x_1} + \gamma' \frac{\partial u}{\partial x_2} \right)(x) = g_2(x) \quad \text{on} \quad \Gamma_2. \]

with Fourier symbols

\[ \sigma_1 = \alpha - \beta t + \gamma \vartheta, \quad \sigma_2 = \alpha' - \beta' t + \gamma' \vartheta \]

where \( t(\xi) = (\xi^2 - k^2)^{1/2} \), \( \vartheta(\xi) = -i\xi \). Both symbols are assumed to be 1-regular, i.e., \( t^{-1} \sigma_j \in GL^\infty \).
The companion operator trick CST 06

Now the trick consists in using a special ansatz (see page 52) where either $\psi_1 = \sigma_2^*$ or $\psi_2 = \sigma_1^*$. This guarantees that $T$ is triangular!

In order to have a strict ansatz, we need that the corresponding BVP with Fourier symbols $\psi_1, \psi_2$ is well-posed. Therefore we consider a companion BVP with

$$B^* = \begin{pmatrix} B_2^* \\ B_2 \end{pmatrix} \quad \text{or} \quad B^* = \begin{pmatrix} B_1 \\ B_1^* \end{pmatrix}.$$ 

In case of the impedance problem it turns out that these problems are explicitly solvable by the DN ansatz, but the resulting potential operator $K$ (generalized inverse to $B^*$) is

- only right invertible with defect number $\beta(K) = 1$, if both $\Im \psi_j$ are negative,
- is strict if at least one $\Im \psi_j$ is positive (corresponding choice),
- needs normalization otherwise or different idea.
Example of a ”bad factorization”  

Thus we found an example (impedance problem with $\beta(\mathcal{K}) = 1$) for

$$L = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$\mathcal{H}^1(Q_1) \xrightarrow{\mathcal{H}} Y$$

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$$

$$T = \begin{pmatrix} T_1 & K_1 \\ K_2 & T_2 \end{pmatrix}$$

where both $\mathcal{K}, T$ are left invertible with defect number 1,

$$T = L\mathcal{K}, \quad L = TK^{-} + LP_1.$$ 

The first is not a full range factorization, but $TK^{-}$ is, and $LP_1$ has rank 1. So it could be proved that $L$ is invertible and the impedance problem also well-posed for these parameters.
Example of a BVP which is not Fredholm but normally solvable

Consider the oblique derivative problem in a 3D rectangular wedge:

\[ Au(x) = (\Delta + k^2)u(x) = 0 \quad \text{in} \quad Q_1 \times \mathbb{R} \subset \mathbb{R}^3 \]
\[ B_1u(x) = \left( \beta \frac{\partial u}{\partial x_2} + \gamma \frac{\partial u}{\partial x_1} \right)(x) = g_1(x) \quad \text{on} \quad \Gamma_1 \times \mathbb{R} \]
\[ B_2u(x) = \left( \beta' \frac{\partial u}{\partial x_1} + \gamma' \frac{\partial u}{\partial x_2} \right)(x) = g_2(x) \quad \text{on} \quad \Gamma_2 \times \mathbb{R} \]

with real coefficients \( \beta, \gamma, \beta', \gamma' \) such that the oblique derivatives are ”directed to the outside”.
10. Conclusion

- Operator factorization and, moreover, operator matrix identities are a very convenient vehicle for the description of equivalence or reduction of problems.
- They are a powerful tool for the inversion of singular operators related to canonical BVPs.
- They enable a precise description of properties of these operators, their normalization and asymptotics.

Many thanks for your attention!
Appendix I: Generalized inverses (1-inverses)

- originated from matrix theory (similar ideas by Fredholm 1903)
  Moore ~ 1906, 1920, Penrose 1950s, Nashed, Nashed-Rall 1976, Ben-Israel and Greville 1974, 2003 (2nd ed.)
- many results are valid for linear operators or ring elements
  Nashed 1976, Nashed-Votruba 1976

**Theorem** Let $T \in \mathcal{L}(X, Y)$ be a bdd. lin. op. in Banach spaces. The following assertions are equivalent:

(i) $TT^-T = T$ for some $T^- \in \mathcal{L}(Y, X)$;
(ii) ker $T$ and im $T$ are complemented (alg. and top.) ;
(iii) There is an operator $T^- \in \mathcal{L}(Y, X)$ such that

$$Tf = g \text{ is solvable iff } TT^- g = g$$

and then, the general solution is given by

$$f = T^-g + (I - T^-T)h, \ h \in Y.$$
Regularity classes of bounded linear operators in Banach spaces (with closed image)

<table>
<thead>
<tr>
<th>$\beta(T) = 0$</th>
<th>$\alpha(T) = 0$</th>
<th>$\alpha(T) &lt; \infty$</th>
<th>$\ker T$ complet.</th>
<th>$\ker T$ closed</th>
</tr>
</thead>
<tbody>
<tr>
<td>bdd. invertible</td>
<td>right inv. Fredholm</td>
<td>right invertible</td>
<td>surjective</td>
<td></td>
</tr>
<tr>
<td>$\beta(T) &lt; \infty$</td>
<td>left inv. Fredholm</td>
<td>Fredholm</td>
<td>right regulariz.</td>
<td>semi-Fred. $\mathcal{F}_-$</td>
</tr>
<tr>
<td>$\text{im} T$ complet.</td>
<td>left invertible</td>
<td>left regulariz.</td>
<td>generalized invertible</td>
<td>no name</td>
</tr>
<tr>
<td>$\text{im} T$ closed</td>
<td>injective</td>
<td>semi-Fred. $\mathcal{F}_+$</td>
<td>no name</td>
<td>normally solvable</td>
</tr>
</tbody>
</table>
Fredholm vs. generalized invertible operators

**Theorem** If $T \in \mathcal{L}(X,Y)$ is a bounded linear operator in Banach spaces, the following assertions are equivalent (see Mikhlin-Prössdorf 80/86, e.g.):

(a) $T \in \mathcal{F}(X,Y)$ is Fredholm

(b) $\exists R_1, R_2 \in \mathcal{L}(Y,X)$ $R_1 T = I_X + V_1$, $T R_2 = I_Y + V_2$, $V_j$ compact

(c) $\exists R_1, R_2 \in \mathcal{L}(Y,X)$ ... such that $V_j$ have finite rank

(d) $\exists R_1, R_2 \in \mathcal{L}(Y,X)$ ... such that $V_j$ are finite rank projectors

(e) $\exists T^- \in \mathcal{L}(Y,X)$ $TT^- T = T$ i.e. $T$ is generalized invertible and $T^- T - I_X$, $TT^- - I_Y$ are finite rank projectors (onto the kernel and along the image of $T$, respectively)

**Remark** The construction of $T^-$ yields an explicit solution.
Appendix II: Sceneries of elliptic BVPs

Wloka 1982/87  "Semi-classical formulation" (scalar PDE)

\[ \Omega \subset \mathbb{R}^n \text{ bounded with } (2m + k, \kappa) - \text{smooth boundary} \]
\[ m \in \mathbb{N}, \ k + \kappa \geq 1 \]

\[ \mathcal{X} = W^{2m+l}_2(\Omega), \ Y_1 = W^l_2(\Omega), \ Y_2 = \prod_{j=1}^{m} W^{2m+l-m_j-1/2}_2(\partial \Omega) \]

\[ A = \sum_{|s| \leq 2m} a_s(x)D^s \text{ uniformly elliptic} \]
\[ \text{with } 2m - \text{smooth coefficients} \]

\[ B_j = \sum_{|s| \leq m_j} b_{j,s}(x)T_0D^s \text{ with Lopatinskii-Shapiro condition} \]
\[ \text{ord } B_j \leq 2m - 1 \text{ and } 2m - \text{smooth coefficients} \]

Main Theorem about equivalence of (a) BVP is elliptic, (b) \( L \) is smoothable, (c) \( L \) is Fredholm, (d) an apriori estimate holds.
Sceneries of elliptic BVPs 2

Hsiao and Wendland 2008

Variational (weak) formulation for

\[ \Omega \in \mathbb{R}^n \text{ strong Lipschitz domain} \]

\[ Au = - \sum_{j,k=1}^n \frac{\partial}{\partial x_j} (a_{jk}(x) \frac{\partial u}{\partial x_k}) + \sum_{j=1}^n b_j(x) \frac{\partial u}{\partial x_j} + c(x)u = f \quad \text{in} \Omega \]

with elliptic symbol and \( f \in \tilde{H}_0^{-1}(\Omega) = \tilde{H}^{-1}(\Omega) \ominus \tilde{H}^{-1}_\Gamma(\Omega). \)

Sesquilinear form:

\[ a_\Omega(u, v) = \int_{\Omega} \left\{ \sum_{j,k=1}^n (a_{jk}(x) \frac{\partial u}{\partial x_k})^\top \frac{\partial \bar{v}}{\partial x_j} + \sum_{j=1}^n (b_j(x) \frac{\partial u}{\partial x_j})^\top \bar{v} + (c(x)u)^\top \bar{v} \right\} \mathrm{d}x \]

Weak solution of the Dirichlet problem (e.g.):

\[ a_\Omega(u, v) = < f, \bar{v} >_{\Omega} \quad \text{for all} \quad v \in H_0^1(\Omega) \]

\[ T_{0,\Gamma}u = g \in H^{1/2}(\Gamma). \]
Further sceneries of elliptic BVPs (working with $L = (A, B)^\top$)

Eskin 1973/81 BVPs for elliptic pseudodifferential equations
Agmon, Agranovich, Boutet de Monvel, Lions, Shamir, Shubin ...

Explicit solution of *canonical problems* in diffraction theory
by operator factorization methods

Meister and Speck 1985-1991 Sommerfeld diffraction problems
Meister, Penzel, Speck, Teixeira 1992-94
Diffraction from rectangular wedges
Castro, Duduchava, Speck, Teixeira 2003-05
Unions of finite intervals, quadrants
Ehrhardt, Nolasco, Speck 2010-12 Non-rectangular wedges, rational angles, conical Riemann surfaces

$$(\Delta + k^2)u = 0 \quad \text{in } \Omega$$

$$T_0(\alpha u + \beta \frac{\partial u}{\partial x} + \gamma \frac{\partial u}{\partial y}) = g \quad \text{on } \Gamma = \partial \Omega$$
Appendix III: Proof of the abstract Babinet principle on p. 19

Proof Putting \( \gamma(\xi_1, \xi_2) = \sqrt{\xi_1^2 + \xi_2^2 + k^2} \), we can show (see MS88)

\[
L_{D,\Omega} \sim W_{\gamma^{-1}, \Sigma} = r_{\Sigma} A_{\gamma^{-1}} = \mathcal{F}^{-1} \gamma^{-1} \cdot \mathcal{F} : H_{\Sigma}^{-1/2} \rightarrow H^{1/2}(\Sigma).
\]

By analogy, the Neumann problem for the complementary screen \( \Sigma_* \) yields an associated operator which satisfies

\[
L_{N,\Omega_*} \sim W_{\gamma, \Sigma_*} = r_{\Sigma_*} A_{\gamma} = \mathcal{F}^{-1} \gamma \cdot \mathcal{F} : H_{\Sigma_*}^{1/2} \rightarrow H^{-1/2}(\Sigma_*).
\]

Composing \( W_{\gamma^{-1}, \Sigma} \) with a continuous extension operator \( \ell_2 : H^{1/2}(\Sigma) \rightarrow H^{1/2}(\mathbb{R}^2) \) we obtain that \( P_2 = \ell_2 r_{\Sigma} \) projects along \( H^{1/2}(\Sigma_*) \) and

\[
W_{\gamma^{-1}, \Sigma} \sim \tilde{W}_{\gamma^{-1}, \Sigma} = \ell_2 W_{\gamma^{-1}, \Sigma} = P_2 A_{\gamma^{-1}}|_{P_1 H^{-1/2}} : H_{\Sigma}^{-1/2} \rightarrow P_2 H^{1/2}.
\]

Similarly

\[
W_{\gamma, \Sigma_*} \sim \tilde{W}_{\gamma, \Sigma_*} = \ell_1 W_{\gamma, \Sigma_*} = Q_1 A_{\gamma}|_{Q_2 H^{-1/2}} : H_{\Sigma_*}^{1/2} \rightarrow Q_1 H^{-1/2}.
\]

Now we have
\begin{itemize}
\item $P_1$ is a projector in $H^{-1/2}(\mathbb{R}^2)$ onto $H^{-1/2}_\Sigma$
\item $P_2$ is a projector in $H^{1/2}(\mathbb{R}^2)$ along $H^{1/2}_{\Sigma^*}$
\item $Q_1 = I - P_1$ is a projector in $H^{-1/2}(\mathbb{R}^2)$ along $H^{-1/2}_\Sigma$
\item $Q_2 = I - P_2$ is a projector in $H^{1/2}(\mathbb{R}^2)$ onto $H^{1/2}_{\Sigma^*}$.
\end{itemize}

\[
\begin{pmatrix}
P_2A_{\gamma^{-1}}P_1 & P_2A_{\gamma^{-1}}Q_1 \\
Q_2A_{\gamma^{-1}}P_1 & Q_2A_{\gamma^{-1}}Q_1
\end{pmatrix}
= \begin{pmatrix}
P_1A_\gamma P_2 & P_1A_\gamma Q_2 \\
Q_1A_\gamma P_2 & Q_1A_\gamma Q_2
\end{pmatrix}^{-1}
\]

because of $A_{\gamma^{-1}} = A_\gamma^{-1}$. Hence $\tilde{W}_{\gamma^{-1},\Sigma}$ and $\tilde{W}_{\gamma,\Sigma^*}$ are matrically coupled, thus equivalent after extension to each other and to $L_{D,\Omega}$ and $L_{N,\Omega^*}$, as well, by transitivity.