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Wiener-Hopf factorization through an intermediate space
Abstract

An operator factorization conception is investigated for a general Wiener-Hopf operator $W = P_2 A|_{P_1 X}$ where $X, Y$ are Banach spaces, $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ are projectors and $A \in \mathcal{L}(X,Y)$ is invertible. Namely we study a particular factorization of $A = A_- CA_+$ where $A_+ : X \to Z$ and $A_- : Z \to Y$ have certain invariance properties and the cross factor $C : Z \to Z$ splits the ”intermediate space” $Z$ into complemented subspaces closely related to the kernel and cokernel of $W$, such that $W$ is equivalent to a ”simpler” operator, $W \sim PC|_{P X}$.

The main result shows equivalence between the generalized invertibility of the Wiener-Hopf operator and this kind of factorization (provided $P_1 \sim P_2$) which implies a formula for a generalized inverse of $W$. It embraces I.B. Simonenko’s generalized factorization of matrix measurable functions in $L^p$ spaces and various other factorization approaches. Various connected theoretical questions are answered such as: How to transform different kinds of factorization into each other? When is $W$ itself the truncation of a cross factor?
General Wiener-Hopf operators

Overall assumptions:

Let $X, Y$ be Banach spaces, $A \in \mathcal{L}(X, Y)$, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ projectors, $Q_1 = I_X - P_1$, $Q_2 = I_Y - P_2$.

Then the operator

$$W = P_2 A|_{P_1 X} = P_1 X \to P_2 Y$$

(1)

is referred to as a general Wiener-Hopf operator (WHO). We shall also call $W$ truncation of the underlying operator $A$. 
Generalized inverses

Given $T \in \mathcal{L}(X, Y)$ an operator $T^- \in \mathcal{L}(Y, X)$ is said to be a generalized inverse of $T$ if $TT^-T = T$. It is called a reflexive generalized inverse if $T^-TT^- = T^-$ holds additionally. In both cases $T$ is said to be generalized invertible (since the existence of a generalized inverse implies the existence of a reflexive generalized inverse, replacing $T^-$ by $T^-TT^-$).

It is well-known that $T$ is generalized invertible if and only if its kernel and image are complemented subspaces of $X$ and $Y$, respectively (which includes the cases of $T$ to be Fredholm, one-sided invertible and others).

The knowledge of a generalized inverse yields solubility conditions and an explicit representation of the general solution of the operator equation $Tf = g$. 
Cross factorization

Let $A$ be boundedly invertible. Then (an operator triple $A_-, C, A_+$ with)

$$A = A_- C A_+$$

(2)

is referred to as a cross factorization of $A$ (with respect to $X, Y, P_1, P_2$), in brief CFn, if the factors $A_\pm$ and $C$ possess the following properties:

$$A_+ \in \mathcal{G}\mathcal{L}(X), \quad A_- \in \mathcal{G}\mathcal{L}(Y),$$

(3)

$$A_+ P_1 X = P_1 X, \quad A_- Q_2 Y = Q_2 Y,$$

and $C \in \mathcal{L}(X, Y)$ splits the spaces $X, Y$ both into four subspaces such that
\[ X = \begin{array}{c}
p_1 X \\ X_1 + X_0 \\
\downarrow \\
Q_1 X \\ X_2 + X_3
\end{array} = \begin{array}{c}
+p \\
C \leftrightarrow
\end{array} \quad (4) \quad \begin{array}{c}
p_2 X \\
Y_1 + Y_2 \\
\downarrow \\
q_2 X \\
Y_0 + Y_3
\end{array}
\]

This means that \( C \) maps each \( X_j \) onto \( Y_j \), \( j = 0, 1, 2, 3 \), i.e., the complemented subspaces \( X_0, X_1, \ldots, Y_3 \) are images of corresponding projectors \( p_0, p_1, \ldots, q_3 \), namely

\[
\begin{align*}
X_1 &= p_1 X = C^{-1}P_2CP_1X, \\
X_0 &= p_0 X = C^{-1}Q_2CP_1X, \\
X_2 &= p_2 X = C^{-1}P_2CQ_1X, \\
X_3 &= p_3 X = C^{-1}Q_2CQ_1X, \\
Y_1 &= q_1 Y = CP_1C^{-1}P_2Y, \\
Y_2 &= q_2 Y = CQ_1C^{-1}P_2Y, \\
Y_2 &= q_1 Y = CP_1C^{-1}Q_2Y, \\
Y_3 &= q_3 Y = CQ_1C^{-1}Q_2Y
\end{align*}
\]

(5)

\( A_\pm \) are called strong WH factors, \( C \) is said to be a cross factor.
The cross factorization theorem \((S\ 1983/85)\)

Let \(A\) be boundedly invertible. Then \(W\) is generalized invertible if and only if a cross factorization of \(A\) exists and, in this case, a formula for a reflexive generalized inverse of \(W\) is given by

\[
W^- = A^{-1}_+ P_1 C^{-1} P_2 A^{-1}_- |_{P_2 Y} : P_2 Y \to P_1 X .
\]

The last part (sufficiency) is proved by inspection whilst the inverse conclusion (necessity) is more complicated. It yields the explicit determination of their kernels and complements of the images provided the factor inverses are known. More consequences on the Fredholm property, explicit presentation of solutions of the equation \(Wf = g\) etc. are immediate.

A most important fact is the equivalence of \(W\) and \(P_2 C|_{P_1 X}\), in brief \(W \sim P_2 C|_{P_1 X}\), namely

\[
W = P_2 A_-|_{P_2 Y} P_2 C|_{P_1 X} P_1 A_+|_{P_1 X} = E P_2 C|_{P_1 X} F
\]

where \(E, F\) are linear homeomorphisms.
WH factorization through an intermediate space

Now we study another type of factorization, which is quite different from the previous and more interesting for many applications.

\[ A = A_- C A_+ \quad : \quad Y \leftarrow Z \leftarrow Z \leftarrow X . \]  

(8)

is referred to as a WH factorization through an intermediate space \( Z \) (with respect to \( X, Y, P_1, P_2 \) (in brief FIS), if the factors \( A_\pm \) and \( C \) possess the following properties: They are linear and boundedly invertible in the above setting with an additional Banach space \( Z \) called intermediate space. Further there is a projector \( P \in \mathcal{L}(Z) \) such that

\[ A_+ P_1 X = P Z , \quad A_- Q Z = Q_2 Y \]  

(9)

with \( Q = I_Z - P \) and such that \( C \in \mathcal{L}(Z) \) splits the space \( Z \) twice into four subspaces:
\[ Z = \begin{array}{c}
\left\{ \begin{array}{c}
PZ \\
X_1 + X_0
\end{array} \right\} \\
\downarrow \\
\left\{ \begin{array}{c}
QZ \\
X_2 + X_3
\end{array} \right\}
\end{array} + \begin{array}{c}
\left\{ \begin{array}{c}
PZ \\
Y_1 + Y_2
\end{array} \right\} \\
\downarrow \\
\left\{ \begin{array}{c}
QZ \\
Y_0 + Y_3
\end{array} \right\}
\end{array} \] (10)

where \( C \) maps each \( X_j \) onto \( Y_j \).

Again \( A_\pm \) are called strong WH factors and \( C \) is said to be a cross factor, now acting from a space \( Z \) into the same space \( Z \).

By analogy to the cross factorization theorem, the following conclusion is straightforward, as well: A FIS of \( A \) implies a reflexive generalized inverse of \( W \) by putting

\[ W^- = A_+^{-1} P C^{-1} P A_-^{-1} \big|_{P_2 Y} : \ P_2 Y \rightarrow P_1 X . \] (11)

The other (necessity) part is not true in general, as we shall see later.
Unbounded FIS

Let $A \in \mathcal{L}(X, Y)$ be boundedly invertible.

A factorization $A = A_- CA_+$ is said to be an unbounded WH factorization through an intermediate (Banach) space $Z$ (unbounded FIS), if the factors $A_{\pm}^1$ are densely defined injective linear operators in the above-mentioned spaces, $P$ and $C \in \mathcal{L}(Z)$ have the same properties as before, and the operator

$$T = A_+^{-1} PC^{-1} PA_-^{-1} : Y \to X$$

(12)

is also densely defined and admits a bounded extension to the full space, in brief $T \in \mathcal{L}(Y, X)$.

I.e., the factorization holds with a cross factor $C \in \mathcal{G}\mathcal{L}(Z)$ and with $A_{\pm}^1$ being densely defined injective linear operators with the above factor properties and with continuous extension in the sense of a FIS.

By analogy, an unbounded cross factorization may be defined.

If $C = I$, the factorization is called canonical.
Full range factorization

Let $T \in \mathcal{L}(X, Y)$ and

$$T = L \ R$$

(13)

where $R \in \mathcal{L}(X, Z)$, $L \in \mathcal{L}(Z, Y)$, $X, Y, Z$ are Banach spaces, $R$ is right invertible and $L$ is left invertible. Then (13) is said to be a full range factorization (FRF) of $T$.

This notion is well-known from matrix theory as full rank factorization ($\dim Z = \text{rank } T$). Evidently a FRF implies that $T^- = R^- L^-$ is a reflexive generalized inverse of $T$ provided $RR^- = I_Z = L^- L$. The intermediate space $Z$ is isomorphic to the image (or range) of $T$ and to any complement of the kernel of $T$, as well.
Main results

Let $A \in \mathcal{L}(X, Y)$ be boundedly invertible, $W = P_2 A|_{P_1 X}$ as before.

**Theorem 1.** $W$ is invertible if and only if $A$ admits a canonical FIS:
\[
A = A_- A_+ \quad (14)
\]
\[
: \quad Y \leftarrow Z \leftarrow X.
\]

**Theorem 2.** The following assertions are equivalent:
(i) $W$ is generalized invertible and $P_1 \sim P_2$ holds,
(ii) $A$ admits a FIS.
Herein the condition $P_1 \sim P_2$ is not redundant.

**Theorem 3.** Every canonical unbounded FIS can be regarded as a canonical bounded FIS, by a change of the intermediate space.
Corollary.

The following statements are equivalent:

(j) $A$ admits a CFn (with respect to $X, Y, P_1, P_2$),

(jj) $W$ admits a full range factorization,

and moreover, if $P_1 \sim P_2$ holds,

(jjj) $A$ admits a FIS (with respect to $X, Y, P_1, P_2$).

In all cases $W$ is generalized invertible, a generalized inverse of $W$ is given in terms of the factorization and a factorization of one type can be computed from any other (via $W^-$).
Example 1: Generalized or $\Phi$-factorization

Let $\Gamma \in \mathbb{C}$ be a closed contour which divides $\mathbb{C} \cup \infty$ into two domains $D_+$ and $D_-$ such that $0 \in D_+$, $\infty \in D_-$ and $\partial D_+ = \partial D_- = \Gamma$. $L^p_\pm \subset L^p(\Gamma)$ ($p > 0$) are the spaces of functions which are boundary values of functions holomorphic in $D_\pm$ in the sense of Privalov (see [LitSpi87] for details). For simplicity we consider the unit circle $\Gamma = \Pi_0 = \{z \in \mathbb{C} : |z| = 1\}$.

A (right, generalized) factorization of $G \in L^\infty(\Gamma)^{n \times n}$ in $L^p$, $1 < p < \infty$, relative to $\Gamma$ is a representation in the form

$$G(z) = G_-(z) \Lambda(z) G_+(z) \quad , \quad z \in \Gamma$$

(15)

where $G_- \in L^p_-$, $G_+ \in L^q_+$, $G_-^{-1} \in L^q_-$, $G_+^{-1} \in L^p_+$, $q = p/(p - 1)$ and the matrix function $\Lambda$ has the form

$$\Lambda(z) = \text{diag} (z^{\kappa_1}, ..., z^{\kappa_n}) \quad , \quad z \in \Gamma$$

(16)

where $z^{\kappa_1} \geq ... \geq z^{\kappa_n}$ are integers.
We investigate here the Toeplitz operator

\[ T = PG \cdot |_{(L^p_+)^n} \quad (17) \]

in the space of vector functions \( X = (L^p_+)^n, p \in ]1, \infty[, \) where \( A = G \cdot \) denotes the multiplication operator and \( P \) the Riesz projection. \( T \) can be seen as an example for a general Wiener-Hopf operator \( W \).

Let us assume that \( G \in L^\infty(\Gamma)^{n \times n} \) admits a (right, generalized) factorization in \( L^p \) [Sim68] for some \( p \in ]1, \infty[ \). Then \( T \) is normally solvable (and moreover Fredholm) if and only if

\[ K = G_+^{-1} \cdot \Lambda Q G_-^{-1} \quad \text{and} \quad K_1 = G_- \cdot PG_-^{-1} \quad (18) \]

are bounded in \( (L^p)^n \). In this case (15) is said to be a \( \Phi \)-factorization of \( G \) [LitSpi87]. For the \( \Phi \)-factorability of \( G \) it is necessary that \( G^{-1} \in (L^\infty)^{n \times n} \).

The \( \Phi \)-factorization can be interpreted in the sense of an unbounded FIS (still with \( X = Y \)) and the formulas for a generalized inverse in terms of the factorization are applicable. We have \( Z = \text{im} A_+ = \text{im} A_-^{-1} \) with the induced norm and \( P, C \in \mathcal{L}(Z) \). This was pointed out already in [CasS95] where the nature of those spaces was studied.
Ex 2: WHOs in diffraction from plane screens

The following species appears particularly in problems of diffraction from plane screens \((n = 2 \text{ or } n = 3)\) such as the Sommerfeld diffraction problem [MS89], its various generalizations, see [CDS6] for instance, and other elliptic boundary value problems [Esk81,HW08,Wlo87]:

\[
W_{\Phi, \Sigma} = r_{\Sigma} \mathcal{F}^{-1} \Phi \cdot \mathcal{F} |_{H_{\Sigma}^r} : H_{\Sigma}^r \to H^s(\Sigma) \tag{19}
\]

where \(\Sigma \subset \mathbb{R}^{n-1}\) is an open set (subset of a hyperplane in \(\mathbb{R}^n\)), \(\mathcal{F}\) the Fourier transformation, \(A = \mathcal{F}^{-1} \Phi \cdot \mathcal{F}\) a translation invariant operator, elliptic of order \(r - s\), i.e., \(\lambda^{s-r} \Phi \in \mathcal{G}L^\infty\) where \(\lambda(\xi) = (|\xi|^2 + 1)^{1/2}\), \(\xi \in \mathbb{R}^{n-1}\), \(r, s \in \mathbb{R}\), and \(H_{\Sigma}^r, H^s(\Sigma)\) Sobolev spaces of distributions supported on \(\Sigma\) or restricted to \(\Sigma\), respectively, see [Esk81] for instance.
The operator (19) is not directly of the form (1) but equivalent, provided \( \text{int clos} \Sigma = \Sigma \) holds and \( \Sigma \) has the strong extension property, i.e., there exists (for any \( s \in \mathbb{R} \)) an extension operator \( E^s_\Sigma \in \mathcal{L}(H^s(\Sigma), H^s(\mathbb{R}^{n-1})) \) which is left invertible by restriction: \( r_\Sigma E^s_\Sigma = I_{H^s(\Sigma)} \). Then we have

\[
W_{\Phi, \Sigma} = r_\Sigma W \sim W = P_2 A|_{P_1 X} \quad (20)
\]

where \( X = H^r, Y = H^s, P_1 \) is a projector in \( H^r \) onto \( H^r_\Sigma \) and \( P_2 \) is a projector in \( H^s \) along \( \Sigma' = \mathbb{R}^{n-1} \setminus \overline{\Sigma} \), for instance \( P_2 = E^s_\Sigma r_\Sigma \).

Concrete examples are known from diffraction theory where also matrix operators with entries like (19) are relevant [CDS14,S14], considered later in Example 6.
Ex 3: Classical WHO's on a half-line
In a simple subclass of (19) we find $n-1 = 1$ and $r = s = 0$, i.e., $X = Y = L^2(\mathbb{R}), \Sigma = \mathbb{R}_+ = ]0, \infty[\}$ and $\Phi \in \mathcal{GC}^\nu(\hat{\mathbb{R}})$ where $\hat{\mathbb{R}}$ denotes the one-point compactification of $\mathbb{R}$. A Wiener-Hopf factorization (in the decomposing algebra $C^\nu(\hat{\mathbb{R}})$) is given by the well-known formulas

$$A = A_- C A_+ = \mathcal{F}^{-1} \Phi_- \cdot \mathcal{F} \quad \mathcal{F}^{-1} \zeta^\kappa \cdot \mathcal{F} \quad \mathcal{F}^{-1} \Phi_+ \cdot \mathcal{F}$$

$$\kappa = \text{ind} \Phi = \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \Phi$$

$$\Phi_\pm = \exp\{P^\pm \log(\zeta^{-\kappa} \Phi)\}$$

where $\kappa \in \mathbb{Z}$ is the winding number of $\Phi$, $\zeta(\xi) = \frac{\xi - i}{\xi + i}$, $\xi \in \mathbb{R}$, $P^\pm = \frac{1}{2}(I \pm H)$ and $H$ is the Hilbert transformation.

The factorization of $A$ in (21) (and of $\Phi$, as well) can be considered as a special case of a CFn (in symmetric space setting $X = Y$) or a FIS (through $Z = X = Y$). The corresponding WHO is always one-sided invertible which in this class is equivalent to be Fredholm, generalized invertible or normally solvable, as the Fourier symbol does not vanish on $\hat{\mathbb{R}}$, cf. Coburn’s Theorem for Toeplitz operators [BoeSil06].
Ex 4: Just a jump at infinity

In applications to diffraction theory we hardly find the preceding class of non-rational Fourier symbols but more likely $\Phi \in \mathcal{GC}^{\nu}(\mathbb{R})$ with a ”jump at infinity”, i.e., being Hölder continuous functions with respect to the two-point compactification of $\mathbb{R}$ which is important for asymptotic results. This is because $\Phi$ belongs to an algebra $\mathcal{A}$ generated by rational functions and a square-root function $\zeta_k^{1/2}(\xi) = \sqrt{\frac{\xi-k}{\xi+k}}$ (originating from the Helmholtz symbol and the lifting process where the wave number $k$ is assumed to have a positive imaginary part). $\mathcal{A}$ is not a decomposing algebra nor an $\mathcal{R}$-algebra [GohKru79,MP86] and therefore $\Phi$ allows only a generalized factorization provided an additional condition at infinity is satisfied, namely

$$\frac{1}{2\pi} \int_{\mathbb{R}} d\arg \Phi + \frac{1}{2} \notin \mathbb{Z}. \quad (22)$$
In this case $\Phi$ can be written as

$$\Phi = \zeta_k^\omega \Psi, \quad \omega = \frac{1}{2\pi i} \int_{\mathbb{R}} d\log \Phi$$ (23)

with $\Psi \in \mathcal{G}C^\omega(\mathbb{R})$ and $\text{ind} \Psi = 0$. The factor $\zeta_k^\omega$ can be replaced by $\zeta^\omega$ (where $k = i$) which has the same behavior at infinity, however can be useful in applications concerning the Helmholtz equation. We write $\zeta^\omega = (\frac{\lambda_-}{\lambda_+})^\omega$ where, for $k, \omega \in \mathbb{C}$ and $\Im mk > 0$,

$$\lambda^\omega_\pm(\xi) = (\xi \pm k)^\omega = \exp\{\omega \log(\xi \pm k)\}, \quad \xi \in \mathbb{C} \setminus \Gamma_k,$$ (24)

with vertical branch cuts $\Gamma_{\mp k}$ ($\Gamma_k = \Gamma_{+k} \cup \Gamma_{-k}$) taken from $\mp k$ to infinity not crossing the real line. Further put

$$\omega = \sigma + i\tau = \kappa + \varepsilon + i\tau$$ (25)

with $\sigma, \kappa, \varepsilon, \tau \in \mathbb{R}, \kappa \in \mathbb{Z}, \varepsilon \in ]-1/2, +1/2[$ provided (22) holds [Dud79,MoST98]. Then we find the factorization

$$\Phi = \Phi_- \zeta^\kappa \Phi_+ = (\Psi_- \lambda_-^{\varepsilon+i\tau}) \zeta^\kappa (\lambda_+^{\varepsilon+i\tau} \Psi_+)$$ (26)

where $\Psi_- \zeta^\kappa \Psi_+$ is a factorization (21) of $\Psi$ like $\Phi$ in the previous example.
It is possible to show that this is a generalized factorization in $L^2(\mathbb{R}, \lambda)$ in the sense of Simonenko. But it is also not hard to verify that it represents a FIS through the weighted space $L^\infty(\mathbb{R}, \lambda^{-\varepsilon})$ which yields that the corresponding factorization of $A$ represents a FIS through the Sobolev space $H^{-\varepsilon}(\mathbb{R}))$. This fact was efficiently used in [MoST98] for developing a normalization method of such operators in the exceptional case of $\varepsilon = 1/2$.

Matrix functions with this kind of elements can lead to factors with logarithmic behavior at infinity and important applications in asymptotic analysis. Further applications appear when tackling WHOs in a half-space $\mathbb{R}^n_+$, see Example 6 later.
**Ex 5: Including oscillating symbols**

Consider $X = Y = L^p(\mathbb{R})^2$, $p \in [1, \infty)$, $P = \chi_+$ (where $\chi_+$ denotes the characteristic function of $\mathbb{R}_+$) as acting in $X$, i.e., $Pf = (\chi_+ f_1, \chi_+ f_2)^T$ for $f = (f_1, f_2)^T \in X$. Further let

$$A = A_{\Phi} = \mathcal{F}^{-1} \Phi \cdot \mathcal{F}$$

$$\Phi = \begin{pmatrix} \tau & 1 \\ 0 & \tau^{-1} \end{pmatrix}$$

(27)

with $\tau(\xi) = e^{i\xi}$, $\xi \in \mathbb{R}$, i.e., where the right/left shift operators appear:

$$A_{\tau \pm 1} = \mathcal{F}^{-1} \tau^{\pm 1} \cdot \mathcal{F}, \quad A_{\tau \pm 1} f_j(x) = f_j(x \mp 1), \quad x \in \mathbb{R}.$$  

The factorization $A = A_- A_+$ with $A_\pm = \mathcal{F}^{-1} \Phi_\pm \cdot \mathcal{F}$,

$$\Phi = \Phi_- \Phi_+ = \begin{pmatrix} 1 & 0 \\ \tau^{-1} & -1 \end{pmatrix} \begin{pmatrix} \tau & 1 \\ 1 & 0 \end{pmatrix}$$

(28)

represents a canonical cross factorization and a canonical FIS with $Z = X$. 
Considering another example, replacing (27) by (cf. [LitSpi87], p.57)

\[
\Phi = \begin{pmatrix} \tau & 0 \\ 1 & \tau^{-1} \end{pmatrix}
\]  

(29)

we obtain, instead of (28) the factorization

\[
\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}
\]  

(30)

which yields a cross factorization (being also a FIS) \( A = A_-CA_+ \) with \( A_- = I \) and a non-trival cross factor \( C \) generating in the diagram (4) the spaces

\[
X_0 = \{0\} \times L^p_{[0,1]}(\mathbb{R})
\]

\[
Y_2 = L^p_{[0,1]}(\mathbb{R}) \times \{0\}
\]

which are isomorphic to the infinite-dimensional defect spaces of \( W \). Hence \( W \) is generalized invertible but neither Fredholm nor semi-Fredholm. A generalized inverse of \( W \) is simply given by formula (6).
In composition with Example 4 one obtains further genuine FIS. The reader may study for instance the case

$$\Phi = \zeta \omega \begin{pmatrix} \tau & 0 \\ 1 & \tau^{-1} \end{pmatrix}$$

combining the ideas of the foregoing examples.

This is one of the simplest cases where the factorization is unbounded, has a nice interpretation as FIS, $W$ is not Fredholm but generalized invertible (with infinite dimensional defect spaces). However still $X = Y$ (a symmetric space setting).
Proof of Theorem 1

Sufficiency. As in many other cases, this step is just a verification of the formula for the inverse. If we have a canonical FIS, then

\[
WW^- = P_2 A^- A_+ |_{P_1 X} A_+^{-1} P A_+^{-1} |_{P_2 Y} = P_2 A^- A_+ P_1 A_+^{-1} P A_+^{-1} |_{P_2 Y} \\
= P_2 A^- P A_+^{-1} |_{P_2 Y} = I |_{P_2 Y}.
\]

Similarly we see that \( W^- W = I |_{P_1 X} \).

Necessity. Let \( W \) be invertible. We identify \( A \) with an equivalent operator matrix and factor this straightforwardly

\[
A \sim \tilde{A} = \begin{pmatrix} P_2 A |_{P_1 X} & P_2 A |_{Q_1 X} \\ Q_2 A |_{P_1 X} & Q_2 A |_{Q_1 X} \end{pmatrix} : P_1 X \times Q_1 X \rightarrow P_2 Y \times Q_2 Y \\
= \begin{pmatrix} I |_{P_2 X} & 0 \\ Q_1 A^{-1} |_{P_2 X} & Q_1 A^{-1} |_{Q_2 X} \end{pmatrix}^{-1} \begin{pmatrix} P_2 A |_{P_1 X} & P_2 A |_{Q_1 X} \\ 0 & I |_{Q_1 X} \end{pmatrix} \\
= \tilde{A}_- \tilde{A}_+ : P_1 X \times Q_1 X \rightarrow P_2 Y \times Q_1 X \rightarrow P_2 Y \times Q_2 Y. \quad (32)
\]
With the above-mentioned identification of the direct sum \( P_1 X + Q_1 X \) and the product space \( P_1 X \times Q_1 X \) (in the algebraic and topological sense) we obtain a factorization of \( A = A_- A_+ \) through \( Z = P_2 Y \times Q_1 X \) because the invertibility of \( W \) implies that (dropping the tildes)

\[
A_+ = \begin{pmatrix}
I_{P_2 Y} & P_2 A|_{Q_1 X} \\
0 & I_{Q_1 X}
\end{pmatrix} \begin{pmatrix}
I_{P_2 Y} A|_{P_1 X} & 0 \\
0 & I_{Q_1 X}
\end{pmatrix}
\]

is invertible. The calculation

\[
A_-^{-1} = A_+ \begin{pmatrix}
P_1 A^{-1}|_{P_2 X} & P_1 A^{-1}|_{Q_2 X} \\
Q_1 A^{-1}|_{P_2 X} & Q_1 A^{-1}|_{Q_2 X}
\end{pmatrix}
= \begin{pmatrix}
I|_{P_2 X} & 0 \\
Q_1 A^{-1}|_{P_2 X} & Q_1 A^{-1}|_{Q_2 X}
\end{pmatrix}
\]

shows that \( A = A_- A_+ \) where \( A_- \) is invertible, as well. Finally, the factor properties of \( A_\pm \) are obvious from the foregoing formulas. \( \square \)

This direct proof, say, has an alternative contained in the following proof, by reduction to a symmetric WHO.
Proof of Theorem 2

**Sufficiency.** If a FIS is given, we define

\[ W^- = A_+^{-1} P C^{-1} P A_-^{-1} |_{P_2 Y} : P_2 Y \to P_1 X. \] (34)

Now we verify \( W W^- W = W \) by calculations similar to the previous and with the help of diagram (4) and by analogy to the calculations in case of a CFn, see [S85], p. 27-29.

The factor properties of \( A_+ \) imply \( P_1 \sim P \) and the factor properties of \( A_- \) imply \( P_2 \sim P \), therefore \( P_1 \sim P_2 \) is necessarily satisfied.
**Necessity.** Since $P_1 \sim P_2$, we can confine ourselves to the symmetric case where $P_2 = P_1$ by splitting an isomorphism from $A$ which maps $P_2 Y$ onto $P_1 X$ and $Q_2 Y$ onto $Q_1 X$. Note that two bounded projectors in Banach spaces are equivalent if and only if their kernels are isomorphic and their co-kernels are isomorphic, as well [BT91]. Hence consider an operator of the form $W = P_1 A|_{P_1 X}$. This can be considered as an element of the form $w = p a p$ in the unital ring $\mathcal{R} = \mathcal{L}(X)$ where $p$ is idempotent and $a$ invertible.

From [S85] we know that the regularity of $w$ [Neu36], i.e., existence of an element $v \in \mathcal{R}$ with $p v p = v$ and $w v w = w$ implies a ring cross factorization given by, e.g.,

\[
\begin{align*}
    a &= a_- c a_+ \\
    &= [e + q a v][a - a v a + w + a(p - v w) a^{-1}(p - w v) a] \\
    &\quad \cdot [e + v a q - (p - v w) a^{-1}(p - w v) a],
\end{align*}
\]

see formula (6.7a) in [S85]. This can be interpreted as a cross factorization of $A$ which coincides with a FIS through the intermediate space $Z = X$ in this symmetric space setting ($X = Y$).
Non-redundance. We give an example where $W$ is generalized invertible, but $A$ does not admit a FIS in a case where $P_1 \sim P_2$ is violated. Let $X = Y = \mathbb{R}^3, P_1 x = (x_1, 0, 0), P_2 = (x_1, x_2, 0)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. If $A \in \mathcal{L}(\mathbb{R}^3)$ is any invertible operator (real linear transformation), $W$ has finite rank (namely rank 0 or 1) and it is therefore generalized invertible, but obviously $P_1 \sim P_2$ is not satisfied, because their kernels are not isomorphic. □
Proof of Theorem 3

Starting with a canonical unbounded FIS $A = A_- A_+$ we put $Z_1 = \text{im } A_+|_{\text{dom } A_+}$ equipped with the norm induced by $X$:

$$\|z\|_{Z_1} = \|A_+^{-1}z\|_X$$

and define $Z$ by taking the closure of $Z_1$ which is obviously isomorphic to the Banach space $X$. Now $Z_2 = \text{im } A_-^{-1}|_{\text{dom } A_-^{-1}}$ yields the same result since $X \cong Y$ because of the assumption that $A$ is invertible. Namely:

$$\|z\|_{Z_1} = \|A_+^{-1}z\|_X \sim \|AA_+^{-1}z\|_Y = \|A_-z\|_Y$$

in the sense of equivalent norms. □
Proof of the Corollary

The only missing step is the conclusion that the generalized invertibility of $T$ yields a full range factorization of $T$. Hence, let $T^{-}$ be a generalized inverse of $T \in \mathcal{L}(X, Y)$ and

$$
\text{Rst } T : X \to \text{im } T
$$

(38)

the image restricted operator, considered as an operator acting not into $Y$ but onto $\text{im } T = TX$. Then

$$
T = TT^{-}T = (TT^{-})|_{\text{im } T} \quad \text{Rst } T
$$

(39)

$$
Y \leftarrow \text{im } T \leftarrow X
$$

represents obviously a full range factorization through $Z = \text{im } T$. Another one would be

$$
T = TT^{-}T = T|_{X_1} \quad \text{Rst } (T^{-}T)
$$

(40)

$$
Y \leftarrow X_1 \leftarrow X
$$

where the intermediate space $X_1 = \text{im } T^{-}$ is a complement of the kernel of $T$. □
Two completion problems

Given a general WHO $W = P_2 A |_{P_1 X}$ one can ask if there is another underlying operator $\tilde{A} \in \mathcal{L}(\tilde{X}, \tilde{Y})$ and two projectors $\tilde{P}_1 \in \mathcal{L}(\tilde{X}), \tilde{P}_2 \in \mathcal{L}(\tilde{Y})$ in suitable Banach spaces such that

$$W = P_2 \tilde{A} |_{\tilde{P}_1 \tilde{X}} = \tilde{P}_1 \tilde{X} \to \tilde{P}_2 \tilde{Y}$$

and such that the new setting is profitable somehow. This question can obviously be seen as a completion problem for an operator matrix

$$\tilde{A} \sim \begin{pmatrix} W & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \end{pmatrix} : \tilde{P}_1 \tilde{X} \times \tilde{Q}_1 \tilde{X} \longrightarrow \tilde{P}_2 \tilde{Y} \times \tilde{Q}_2 \tilde{Y}. \quad (41)$$

**Problem 1.** Given a WHO where $A$ is not invertible, look for a setting $\tilde{X}, \tilde{Y}, \tilde{A}, \tilde{P}_1, \tilde{P}_2$ such that $W = \tilde{W} = \tilde{P}_2 \tilde{A} |_{\tilde{P}_1 \tilde{X}}$ where $\tilde{A}$ is boundedly invertible.

**Problem 2.** Given a WHO where $A$ is invertible, look for a setting such that $W = \tilde{W} = \tilde{P}_2 \tilde{A} |_{\tilde{P}_1 \tilde{X}}$ where $\tilde{A}$ is a cross factor.
Solution of Problem 1

Any bounded linear operator acting in Banach spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, can be considered as a general WHO $T = P_2 A|_{P_1 X}$ where the underlying operator $A$ is boundedly invertible.

**Proof** We consider the topological product spaces $X = \mathcal{X} \times \mathcal{Y}$, $Y = \mathcal{Y} \times \mathcal{X}$ as Banach spaces and

$$A = \begin{pmatrix} T & \mu I_Y \\ \mu I_X & 0 \end{pmatrix} : X \to Y \quad (42)$$

where $\mu \in \mathbb{C}, |\mu| > \|T\|$, and (interpreting the zeroes appropriately)

$$P_1 : X \to \mathcal{X} \times \{0\} \quad , \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$P_2 : Y \to \mathcal{Y} \times \{0\} \quad , \quad \begin{pmatrix} y \\ x \end{pmatrix} \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}.$$ 

Then we have an invertible $A$ and $T$ identified with

$$P_2 A|_{P_1 X} : \begin{pmatrix} x \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} Tx \\ \mu x \end{pmatrix} \mapsto \begin{pmatrix} Tx \\ 0 \end{pmatrix} \quad \square$$
Solution of Problem 2

Let \( W = P_2 A|_{P_1 X} \) be given as in Definition 1.1, \( A \) being invertible and \( W \) generalized invertible. Then \( W \) can be considered as a truncation of a cross factor, acting between the same spaces as \( A \) does:

\[
W = P_2 \ C|_{P_1 X}. \tag{43}
\]

**Proof.** In the symmetric case, this is an interpretation of formula (35). Namely, if \( v \) is a reflexive generalized inverse of \( w \), one can verify in (35) that \( p a p = p c p \), since \( p a_- p = p \) and \( p a_+ p = p \), as well.

In the asymmetric case we modify Formula (35) in the sense of (32):

\[
\tilde{A} = \tilde{A}_- \ C \ \tilde{A}_+ : \ P_1 X \times Q_1 X \to P_2 Y \times Q_2 Y \tag{44}
\]

\[
\tilde{A}_- = \begin{pmatrix}
I|_{P_2 Y} & 0 \\
Q_2 A P_1 V|_{P_2 Y} & I|_{Q_2 Y}
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
W & P_2 (A - AVP_2 A)|_{Q_1 X} \\
Q_2 (A - AVP_2 A)|_{P_1 X} & Q_2 (A - AVP_2 A + A (P_1 - VW P_1) A^{-1} (P_2 - WPV_2 A)|_{Q_1 X}
\end{pmatrix}
\]

\[
\tilde{A}_+ = \begin{pmatrix}
I|_{P_1 X} & (VP_2 A - (P_1 - VW P_1) A^{-1} (P_2 - WPV_2 A)|_{Q_1 X} \\
0 & I|_{Q_1 X}
\end{pmatrix}
\]

Verification is carried out by analogy to the symmetric case. \( \square \)
Ex 6: Interface problems in $\mathbb{R}^2$

An important variant of Example 2 is the WHO

$$W = r_+ A|_{P_1X} : H^{1/2}_+ \times H^{-1/2}_+ \to H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+)$$

where $X = Y = H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$, $\Sigma = \mathbb{R}_+ = ]0, \infty[$, $r_+ = r_{\mathbb{R}_+}$ and $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ with

$$\sigma_\lambda = \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix}$$

and $t(\xi) = (\xi^2 - k^2)^{1/2}$, $\xi \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

A canonical generalized factorization of $\sigma_\lambda = \sigma_\lambda_- \sigma_\lambda_+$ was derived with the help of Khrapkov’s formulas and Daniele’s trick [S89] (stimulated by the mixed Dirichlet-Neumann problem where $\lambda = 1$ [Raw81]):
\[ \sigma_{\lambda^+} = (1 - \lambda^{-1})^{-1/4} \begin{pmatrix} c_+ & -s_+ \sqrt{\lambda}/t \\ -c_+ \xi/\sqrt{\lambda} - s_+ t/\sqrt{\lambda} & s_+ \xi/t + c_+ \end{pmatrix} \]
\[ \sigma_{\lambda^-} = (1 - \lambda^{-1})^{-1/4} \begin{pmatrix} c_- - s_- \xi/t & -s_- \sqrt{\lambda}/t \\ -s_- t/\sqrt{\lambda} + c_- \xi/\sqrt{\lambda} & c_- \end{pmatrix} \]

where

\[ c_{\pm}(\xi) = \cosh[C \log \gamma_{\pm}(\xi)] \]
\[ s_{\pm}(\xi) = \sinh[C \log \gamma_{\pm}(\xi)] \]
\[ \gamma_{\pm}(\xi) = \frac{\sqrt{k \pm \xi} + i \sqrt{k \mp \xi}}{\sqrt{2k}} , \quad \xi \in \mathbb{R} \]
\[ C = \frac{i}{\pi} \log \frac{\sqrt{\lambda} + 1}{\sqrt{\lambda} - 1} . \]
Because of the asymptotic behavior of $\sigma_{\lambda \pm}$ at infinity, the corresponding factorization of $A = \mathcal{F}^{-1} \sigma_{\lambda} \cdot \mathcal{F}$ represents a canonical Wiener-Hopf factorization through a vector Sobolev space:

$$
A_{\lambda} = A_{\lambda -} A_{\lambda +} = \mathcal{F}^{-1} \sigma_{\lambda -} \cdot \mathcal{F} \quad \mathcal{F}^{-1} \sigma_{\lambda +} \cdot \mathcal{F}
$$

$$
H^{1/2} \times H^{-1/2} \leftarrow Z \leftarrow H^{1/2} \times H^{-1/2}
$$

(48)

$$
Z = H^\vartheta(\mathbb{R}) \quad , \quad \vartheta = (\vartheta_1, \vartheta_2) = \left( \frac{1}{2}(1 - \delta), \frac{1}{2}(\delta - 1) \right)
$$

where $\delta = \Re C = \frac{-1}{\pi} \arg \frac{\sqrt{\lambda} + 1}{\sqrt{\lambda} - 1} \in ]0, 1]$. 
Ex 6 ctd: Interface problems in $\mathbb{R}^n$, $n \geq 3$

Consider the higher-dimensional case ($m = n - 1 \geq 2$) where $\Sigma$ is a half-space which is of particular interest in various applications:

$$X = Y = H^{1/2}(\mathbb{R}^m) \times H^{-1/2}(\mathbb{R}^m), \quad \Sigma = \mathbb{R}^m_+ = \mathbb{R}^{m-1} \times [0, \infty[$$

and $t(\xi) = (\xi_1^2 + \ldots + \xi_m^2 - k^2)^{1/2}$, $\xi = (\xi', \xi_m) \in \mathbb{R}^m$, we can consider the same factorization given by (47) replacing $k$ by $(k^2 - \xi'^2)^{1/2}$, i.e., the previous factorization as to be parameter-dependent of $\xi' \in \mathbb{R}^{m-1}$. It turns out that the factorization can be seen as a canonical FIS of $A$ where the intermediate space is an unisotropic vector Sobolev space

$$Z = H^\varrho(\mathbb{R}^m) \times H^{-\varrho}(\mathbb{R}^m)$$

$$H^\varrho(\mathbb{R}^m) = \mathcal{F}(w_{\varrho}L^2(\mathbb{R}^m)), \quad w_{\varrho}(\xi) = (1 + |\xi'|^2)^{\varrho_1/2}(1 + \xi_m^2)^{\varrho_2/2}$$

$$\varrho = (\varrho_1, \varrho_2) = \left(\frac{1}{2}(\delta - 1), \frac{1}{2}(1 - \delta)\right),$$

see a forthcoming paper [S14] for more details.
Constructing a FRF of $W$ from a FIS of $A$

We study the question: How can a FIS be employed to construct a FRF of $W$ in a more direct (constructive) way than via a generalized inverse? In general the construction of a FRF of $W$ is a difficult task and not much treated in the literature, see [S83] where a so-called weak factorization was used and the complicated interaction between the two factors was pointed out.

Looking at the symmetric situation $W = PA|_{PX}$, $A \in \mathcal{GL}(X)$, $P^2 = P \in \mathcal{L}(X)$, a weak factorization $A = B_-B_+$ is characterized by

$$B_\pm \in \mathcal{GL}(X) \ , \ B_+P = PB_+P \ , \ PB_- = PB_-P ,$$

i.e., $B_+$ maps $PX$ into $PX$ and $B_-$ maps the complement $QX$ into $QX$. This yields

$$W = PB_-B_+|_{PX} = PB_-|_{PX} PB_+|_{PX} = W_- W_+$$

where $W_-$ is right invertible and $W_+$ is left invertible. I.e., we do not have a FRF and the consequences in general are poor.
However, in more special situations, the two operators $W_-$ and $W_+$ commute. It happens typically in the case of classical Toeplitz and Wiener-Hopf operators. Looking again at Example 1 we observe that the (reduced) WHO
\[
T = PC|_{PX} = P \text{ diag} \left( z^{\kappa_1}, ..., z^{\kappa_n} \right)|_{PX}
\]
has also this property: Writing
\[
T = T_- T_+ = P \text{ diag} \left( z^{\kappa^-_1}, ..., z^{\kappa^-_n} \right)|_{PX} P \text{ diag} \left( z^{\kappa^+_1}, ..., z^{\kappa^+_n} \right)|_{PX}
\]
where $\kappa^+_j = \max\{\kappa_j, 0\}$, $\kappa^-_j = \min\{\kappa_j, 0\}$, we see that $T_-$ and $T_+$ commute. So we arrive at the conclusion that any $\Phi$-factorization of a measurable matrix function can be easily transformed into a FRF of $W$, which can be also seen as a consequence of the general version:
**Corollary.** Let $W$ be given as before and $A = A_- CA_+$ be a FIS where $PC_j|_{PX} = \text{diag}(T_1, \ldots, T_n)$ and all $T_j$ are one-sided invertible. Then a FRF of $W$ is given by

$$W = (P_2 A_- C_+|_{PZ})(PC_- A_+|_{P_1 X})$$

with $PC_+|_{PZ}$ right invertible and $PC_+|_{PZ}$ left invertible. Note that the knowledge of a CFn instead of a FIS does not suffice because the commutativity of the two middle factors is needed.
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References


