Homotopical Algebra of Coisotropic Submanifolds

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1. Coisotropic Submanifolds

Let M be a smooth finite-dimensional manifold. The graded algebra of multivector fields $\mathcal{V}(M)$ on M carries the structure of a Gerstenhaber-algebra; the bracket being the Schouten-Nijenhuis bracket $[-,-]_{SN}$. A Poisson bivector field Π on M is a Maurer-Cartan element of $(\mathcal{V}(M)[1],[-,-]_{SN})$, i.e. a bivector field Π satisfying $[\Pi,\Pi]_{SN}=0$. Let $\Pi^{\#}$ denote the bundle map between T^*M and TM given by contraction with Π.

Consider a submanifold S of M. The conormal bundle N^*S of S is defined to be the annihilator of TS inside T^*M.

Definition. *A submanifold S of a smooth finite-dimensional Poisson manifold (M,Π) is called coisotropic if the restriction of $\Pi^{\#}$ to N^*S has image in TS.*

Any coisotropic submanifold S is equipped with a natural foliation $\mathcal{F}_S := \Pi^{\#}(N^*S)$ which is involutive and hence integrable. We denote the corresponding leaf space by $\underline{S} := S/\sim_{\mathcal{F}_S}$. This space is usually very ill-behaved (non-smooth, non-Hausdorff, etc.). One defines

$$\mathcal{C}^\infty(\underline{S}) := \{f \in \mathcal{C}^\infty(S) : X(f)=0 \text{ for all } X \in \Gamma(\mathcal{F}_S)\}.$$

This is a subalgebra of $\mathcal{C}^\infty(S)$ and comes quipped with a Poisson bracket $\{-,-\}_{\underline{S}}$ inherited from (M,Π).

The easiest way to describe the Lie algboid structure on N^*S is via the associated Lie algebroid cocomplex. Define the projection $Pr : \mathcal{V}(M) \to \Gamma(\bigwedge NS)$ to be the unique algebra morphism extending the restriction $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(S)$ and projection $\Gamma(TM) \to \Gamma(NS)$. The Lie algebroid cocomplex is $\Gamma(\bigwedge NS)$ equipped with the differential given by

$$\partial_S(X) := Pr([\Pi,\tilde{X}]_{SN})$$

where $\tilde{X}$ is any extension of $X \in \Gamma(\bigwedge NS)$ to a multivector field on M. The cohomology of the cocomplex $(\Gamma(\bigwedge NS),\partial_S)$ is called the Lie algebroid cohomology of S.

2. Enriched Lie Algbroid cocomplex

We describe a homological resolution of the Poisson algebra $(\mathcal{C}^\infty(\underline{S}),\{-,-\}_{\underline{S}})$ in terms of an L_∞-algebra structure on $(\Gamma(\bigwedge NS),\partial_S)$. The degree zero component of the cohomology $H(\Gamma(\bigwedge NS),\partial_S)$ is isomorphic to $\mathcal{C}^\infty(\underline{S})$ and the induced bracket coincides with $\{-,-\}_{\underline{S}}$. This structure was found by Oh and Park in [OP] and it can also be derived as the classical limit of the Poisson Sigma model with boundary conditions given by S ([CF]).

Assume for the moment that M is the total space of a vector bundle $E \to S$. Then there is a natural section ι of $Pr : \mathcal{V}(M) \to \Gamma(\bigwedge NS)$ and generalizing the formual for the Lie algebroid differential ∂_s one can define higher structure maps $\mu_n : (\Gamma(\bigwedge NS))^{\otimes n} \to \Gamma(\bigwedge NS)[2-n]$ by

$$X_1 \otimes \cdots \otimes X_n \mapsto (-1)^{\#} Pr([[\cdots[\Pi,\iota(X_1)]_{SN},\cdots]_{SN},\iota(X_n)]_{SN}).$$

This is an example of the derived brackets formalism as presented in [V]: the input data are 1. a graded Lie algebra $\mathfrak{h}$ $(\mathcal{V}(E)[1],[-,-]_{SN})$, 2. a splitting of $\mathfrak{h}$ into an abelian part $\mathfrak{a}$ $(\Gamma(\bigwedge NS)[1])$ and a Lie subalgebra and 3. a Maurer-Cartan element γ (Π) of $\mathfrak{h}$, i.e. $\gamma \in \mathfrak{h}^1$ satisfying $[\gamma,\gamma]=0$. The main theorem is that the higher structure maps as defined above satisfy the defining relations of an L_∞-algebra, i.e. the following family of Jacobiators vanishes identically:

$$J^n(x_1\cdots x_n) := \sum_{r+s=n} (-1)^{\#}\mu_{s+1}(\mu_r(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)}).$$

Consequently:

Theorem. $(\Gamma(\bigwedge NS),\partial_S=\mu_1,\mu_2,\cdots)$ *is an L_∞-algebra extending the Lie algebroid cocomplex associated to S.*

For arbitrary coisotropic submanifold $S \hookrightarrow M$ one chooses an embedding $NS \hookrightarrow M$ and replaces M by the image E of this embedding. By definition E is a vector bundle over S. Hence one can define higher structure maps as before and proves:

Theorem. *Two choices of embeddings $NS \hookrightarrow M$ lead to L_∞-isomorphic L_∞-algebra structures on $(\Gamma(\bigwedge NS),\partial_S)$.*

In the symplectic setting both theorems were first proven in [OP]. The connection to the derived brackets formalism was made explicite in [CF]. The second theorem can be concluded from a general invariance result on the derived brackets formalism presented in [CS].

3. BFV-complex

There is an alternative resolution of $(\mathcal{C}^\infty(\underline{S}),\{-,-\}_{\underline{S}})$ which originated from physical considerations concerning classical systems with symmetries ([BF], [BV]). Later this approach known as the BFV-complex (short for Batalin-Vilkovisky-Fradkin complex) was reinterpreted in terms of homological algebra ([St]). The globalization used in the smooth setting is due to [B] and [He].

Again one fixes an embedding $NS \hookrightarrow M$ so that one can assume that M is the total space of a vector bundle $p : E \to S$. The bundle $\mathcal{E}$ is the pull back bundle of $E \to S$ under p and $BFV(S) := \Gamma(\bigwedge(\mathcal{E}\oplus\mathcal{E}^*) \to E)$. Set $\mathcal{A} := \bigwedge_{BFV(S)} Der(BFV(S))$ and observe that it carries the structure of a Gerstenhaber algebra with a bracket resembling the Schouten-Nijenhuis bracket. We denote this bracket by $[-,-]_{SN}$ as before. There is a Maurer-Cartan element G in $\mathcal{A}$ given by the fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$ (extented as a derivation). Hence $(\mathcal{A}[1],[G,-]_{SN},[-,-]_{SN})$ is a differential graded Lie algebra. In [Sch2] we prove the following:

Theorem. *The differential graded Lie algebra $(\mathcal{A}[1],[G,-]_{SN},[-,-]_{SN})$ is L_∞ quasi-isomorphic to the graded Lie algebra $(\mathcal{V}(E)[1],[-,-]_{SN})$.*

Consequently every Poisson bivector field Π on E (i.e. a Maurer-Cartan element of $(\mathcal{V}[1],[-,-]_{SN})$) can be lifted to a Maurer-Cartan element $\tilde{\Pi}$ of $\mathcal{A}[1]$. Setting $\hat{\Pi} := \tilde{\Pi} + G$ yields a Maurer-Cartan element of $(\mathcal{A}[1],[-,-]_{SN})$ and applying the derived bracket formalism equips $BFV(S)$ with the structure of a (graded) Poisson algebra $(BFV(S),[-,-]_{BFV})$. A less conceptual construction of $[-,-]_{BFV}$ was known from [R] in the symplectic setting and extented to the Poisson setting in [He].

The next step is to construct a particular element $\Omega \in BFV(S)$ such that $[\Omega,\Omega]_{BFV}=0$. One starts with $\Omega_0 \in BFV(S)$ given by the tautological section of $\mathcal{E} \to E$ and computes $[\Omega_0,\Omega_0]_{BFV}$ which is non-zero. However – using homological perturbation theory – one can prove that there are appropriate corrections $\Omega_1,\Omega_2,\ldots$ and a bounded decreasing filtration on $BFV(S)$ such that $\lim_{n\to\infty}[\Omega_0+\cdots+\Omega_n,\Omega_0+\cdots+\Omega_n]_{BFV}=0$. By boundedness of the filtration the limit $\lim_{n\to\infty}\Omega_0+\cdots\Omega_n =: \Omega$ is actually a finite sum. This strategy is essentially contained in [St] and can be extented to the smooth setting.

The BFV-complex is the differential graded Poisson algebra $(BFV(S),[\Omega,-]_{BFV},[-,-]_{BFV})$. It is well-known that $H(BFV(S),[\Omega,-]_{BFV})$ is isomorphic to the cohomology of the Lie algebroid cocomplex $(\Gamma(\bigwedge NS),\partial_S)$.

A considerable extension is the following

Theorem. *The differential graded Lie algebra $(BFV(S),[\Omega,-]_{BFV},[-,-]_{BFV})$ is L_∞ quasi-isomorphic to the L_∞-algebra $(\Gamma(\bigwedge NS),\partial_S,\mu_2,\mu_3,\ldots)$.*

This is the main result of [Sch1]. It says that the two resolutions essentially contain the same information as far as homotopical algebra is concerned. Furthermore by using the refined construction of the BFV-complex outlined before one obtains ([Sch2]):

Theorem. *All choices involved in the construction of $(BFV(S),[\Omega,-]_{BFV},[-,-]_{BFV})$ (that is 1. an embedding $NS \hookrightarrow M$, 2. an affine connection on $NS \to S$ and 3. Ω) lead to isomorphic differential graded Poisson algebras if one works in the formal setting.*

Here formal refers to the fact that one has to replace NS by a formal neighbourhood of S inside M for the Theorem to work. While different choices of connection or Ω do not cause probles, different choices of embeddings $NS \hookrightarrow M$ can lead to drastic changes otherwise.

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This work has been partially supported by SNF Grant #20-113439, by the European Union through the FP6 Marie Curie RTN ENIGMA (contract number MRTN-CT-2004-5652), and the European Science Foundation through the MISGAM program. Moreover the author was supported by the ESI for Mathematical Physics in Vienna during his stay in July and August 2007.