Coisotropic Submanifolds and the BFV-Complex

Florian Schätz

April 27th, 2009

Coisotropic submanifolds: submanifolds of *Poisson manifolds*

Definition (*Poisson Manifolds*)

- *Poisson manifold*: manifold M & Poisson bivector field π,
- *Poisson bivector field*: $\pi \in \Gamma(\wedge^2 TM)$ satisfying integrability-condition,
- *Integrability condition*:

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M) \\ (f,g) &\mapsto \pi(f,g) \end{aligned}$$

Lie bracket on $\mathcal{C}^\infty(M)$, i.e.

$$\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$$

$\forall$ smooth functions f, g, h.

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Examples of Poisson manifolds

- $(\mathbb{R}^n \oplus \mathbb{R}^n, \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} + \cdots + \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n}) \rightsquigarrow$
$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial x^i} \right),$$
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(M, π) Poisson manifold; S submanifold;

- *vanishing ideal* $\mathcal{I}(S)$ of S in M is

$$\mathcal{I}(S) := \{f \in \mathcal{C}^\infty(M) : f|_S = 0\}.$$

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- $\mathbb{R}^n \oplus 0$ and $0 \oplus \mathbb{R}$ in $(\mathbb{R}^n \oplus \mathbb{R}^n, \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} + \cdots + \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial y^n})$,
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Good description of $\{S$ coisotropic submanifold of $(M, \pi)\}$? Properties?

- group of diffeomorphisms acts on $\{S$ submanifold of $M\}$,
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questions too hard! $\leadsto$ simplify...

fix S coisotropic, study questions only "near" S!

- "linearize" M near $S \leadsto$ assume: M total space of a vector bundle $E \to S$,
- $\mu \in \Gamma(E)$ *coisotropic* $:\Leftrightarrow$ graph(μ) coisotropic submanifold,
- $\mathcal{C}(E, \pi)$ set of coisotropic sections.

Q1) How to describe $\mathcal{C}(E, \pi)$? What are its properties?

- action of Hamiltonian diffeomorphisms $\leadsto$ equivalence relation $\sim_H$ on $\mathcal{C}(E, \pi)$,
- $\mathcal{M}(E, \pi) :=$ equivalence classes of $\sim_H$.

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$$\mathbb{R} \oplus 0 \hookrightarrow (\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$$

- $\mathcal{C}(\mathbb{R}^2 \to \mathbb{R} \oplus 0, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \mathcal{C}^\infty(\mathbb{R})$,
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L Lagrangian submanifold, (Darboux-Weinstein) $\Rightarrow$ suffices to consider $L \hookrightarrow (T^*L, \omega_{\text{can}})$ [universal model].

- *graph of $\mu : L \to T^*L$ is Lagrangian $\Leftrightarrow$ μ is closed as a one-form on L;*

$$\mathcal{C}(T^*L \to L, \omega_{\text{can}}) = \{\mu \in \Omega^1(L) : d_{DR}(\mu) = 0\},$$

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- answers Q1) and Q2) for Lagrangian submanifolds,
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L Lagrangian submanifold, (Darboux-Weinstein) $\Rightarrow$ suffices to consider $L \hookrightarrow (T^*L, \omega_{\text{can}})$ [universal model].

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Right replacement of $(\Omega^\bullet(L), d_{DR})$?

- $S \hookrightarrow (E, \pi)$ coisotropic $\rightsquigarrow$ $(\Gamma(\wedge E), \partial_\pi)$, *Lie algebroid complex*
- for L Lagrangian this complex isomorphic to $(\Omega^\bullet(L), d_{DR})$,
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Lie Algebroid Complex

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$$\mathbb{R} \oplus 0 \hookrightarrow (\mathbb{R}^2, (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$$

- $\mathcal{C}(\mathbb{R}^2 \to \mathbb{R} \oplus 0, (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \cong \mathcal{C}^\infty(\mathbb{R})$,
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$$\text{isomorphism induced from} \quad f \mapsto \begin{cases} + & f(0) > 0, \\ f'(0) & f(0) = 0, \\ - & f(0) < 0. \end{cases}$$

Homological prediction:

Lie algebroid complex $K^\bullet \cong (\mathcal{C}^\infty(\mathbb{R})[0] \xrightarrow{x^2(-)} \mathcal{C}^\infty(\mathbb{R})[-1]) \quad \Rightarrow$

- $\ker(\mathcal{C}^\infty(\mathbb{R})[-1] \xrightarrow{0} 0) = \mathcal{C}^\infty(\mathbb{R})$,
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S coisotropic submanifold of (E, π), $E \to S$ vector bundle;

Oh/Park & Cattaneo/Felder: *higher order operations*, i.e.

$$\lambda_k : \Gamma(\wedge^{i_1} E) \times \cdots \times \Gamma(\wedge^{i_k} E) \to \Gamma(\wedge^{i_1 + \cdots + i_k + 2 - k} E).$$

Properties:

- $\lambda_1 = \partial_\pi$,
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Example II, revisited

Recall:

- consider $\mathbb{R} \oplus 0 \hookrightarrow (\mathbb{R}^2, (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$;
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Higher order operations $\rightsquigarrow$

$$\begin{aligned} \mathcal{C}^\infty(\mathbb{R})[0] \times \mathcal{C}^\infty(\mathbb{R})[-1] \times \mathcal{C}^\infty(\mathbb{R})[-1] &\to \mathcal{C}^\infty(\mathbb{R})[-1] \\ (f, g, h) &\mapsto \left(\frac{df}{dx}\right) gh. \end{aligned}$$

What to do with this piece of data?

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Aim: complex $(C^\bullet, d) := (\cdots \to C^k \xrightarrow{d^k} C^{k+1} \to \cdots)$;

How to interpret $H^1(C^\bullet, d)$ “geometrically”?

- *groupoid*: category all of whose morphisms are invertible,
- groupoid attached to $(C^\bullet, d)$:
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Geometric interpretation of $H^1(-)$

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Maurer–Cartan elements and gauge-action

- this interpretation of $H^1(C, d)$ extends to complexes enriched by higher order operations!
- kernel of $d^1 : C^1 \to C^2$ replaced by *Maurer–Cartan elements*,
- action

$$\begin{aligned} C^0 \times \ker(d^1 : C^1 \to C^2) &\to \ker(d^1 : C^1 \to C^2), \\ (v, x) &\mapsto x + d^0(v) \end{aligned}$$

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- *convergence issues*: higher order operations $(\lambda_k)_{k\geq 1}$ on $\Gamma(\wedge E)$ can be nontrivial for infinitly many $k \geq 1$,
- $\rightsquigarrow$ need some *completion* to define Maurer–Cartan elements and gauge-action,
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- $\rightsquigarrow$ homotopy Lie algebroid cannot distinguish

$$\{0\} \hookrightarrow (\mathbb{R}^2, 0) \quad \text{from} \quad \{0\} \hookrightarrow (\mathbb{R}^2, e^{\left(-\frac{1}{x^2+y^2}\right)} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})!$$

Are we done?!

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- $H^\bullet(\Gamma(\wedge E), \partial_\pi) \cong H^\bullet(BFV(E,\pi), D)$,
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Theorem (*S.*)

The homotopy Lie algebroid and the BFV-complex are L_∞ quasi-isomorphic.

Remark:

- structures cannot be isomorphic as L_∞-algebras,
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More structure:

- topology on $\Gamma(E)$ induces one on $\mathcal{C}(E, \pi)$ and $\mathcal{M}(E, \pi)$, *when is* $\mathcal{M}(E, \pi)$ *Hausdorff, locally path-connected,...?*
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