INSTITUTO SUPERIOR TÉCNICO
Licenciatura em Matemática Aplicada e Computação

MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 17 de Julho de 2010

[1][20]

$$\delta_{\tilde z_1} = -\frac{x}{z_1}\,\delta_{\tilde x} + \delta_1$$

$$\delta_{\tilde z_2} = \frac{x}{z_2}\,\delta_{\tilde x} + \delta_2$$

$$\delta_{\tilde z_3} = -\delta_{\tilde z_1} + \delta_3 = \frac{x}{z_1}\,\delta_{\tilde x} - \delta_1 + \delta_3$$

$$\delta_{\tilde z_4} = -\delta_{\tilde z_2} + \delta_4 = -\frac{x}{z_2}\,\delta_{\tilde x} - \delta_2 + \delta_4$$

$$\begin{aligned}
\delta_{\tilde z} &= \frac{z_3}{z}\,\delta_{\tilde z_3} - \frac{z_4}{z}\,\delta_{\tilde z_3} + \delta_5 \\
&= \frac{x}{z}\left(\frac{z_3}{z_1} + \frac{z_4}{z_2}\right)\delta_{\tilde x} + \frac{z_3}{z}\,(\delta_3 - \delta_1) + \frac{z_4}{z}\,(\delta_2 - \delta_4) + \delta_5 \\
&= \frac{1+x^2}{1-x^2}\,\delta_{\tilde x} + \frac{1+x}{2x}(\delta_3 - \delta_1) + \frac{1-x}{2x}(\delta_2 - \delta_4) + \delta_5 \\
&=: p_f(x)\,\delta_{\tilde x} + q_{31}(x)(\delta_3 - \delta_1) + q_{24}(x)(\delta_2 - \delta_4) + \delta_5
\end{aligned}$$

O problema é estável ou bem posto excepto para $x \approx 1$ e $x \approx -1$ pois $p_f(x)$ é singular para $x = \pm 1$.

O algoritmo para o cálculo de $f(x)$ é numericamente instável para os mesmos valores de x e ainda para $x \approx 0$ pois $q_{31}(x)$ e $q_{24}(x)$ são singulares para $x = 0$.

[2]
(a)[15]

A função $g : \mathbb{R}^2 \to \mathbb{R}^2$ tem um único ponto fixo em D pois são satisfeitas as condições requeridas pelo teorema do ponto fixo. Com efeito:

(i) $g \in C^1(D)$

(ii) $$J_g(x) = \begin{bmatrix} -\rho \sin(x_1 - x_2) & \rho \sin(x_1 - x_2) \\ -\rho \cos(x_1 + x_2) & -\rho \cos(x_1 + x_2) \end{bmatrix}$$

$$\|J_g(x)\|_1 = \rho \left(|\sin(x_1 - x_2)| + |\cos(x_1 + x_2)| \right)$$

$$\sup_{x \in D} \|J_g(x)\|_1 \leq 2\rho =: L < 1$$

(iii) $g_1(x) \in [-\rho, \rho] \subset [-1, 1] \quad \forall x \in D$

$$g_2(x) \in [0, 2\rho] \subset [-1, 1] \quad \forall x \in D$$

$$\implies \quad g(D) \subset D$$

A função $g : \mathbb{R}^2 \to \mathbb{R}^2$ tem um único ponto fixo em $\mathbb{R}^2$ pois são igualmente satisfeitas as condições requeridas pelo teorema do ponto fixo em qualquer conjunto $X = [a_1, a_2] \times [b_1, b_2]$ que contenha D. Basta notar que

$$g \in C^1(X), \qquad \sup_{x \in X} \|J_g(x)\|_1 \leq 2\rho < 1, \qquad g(X) \subset D \subset X.$$

(b)[15]

$$x^{(0)} = [0 \;\; 0]^T \qquad x^{(1)} = g\left(x^{(0)}\right) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T$$

$$\|z - x^{(m)}\|_1 \leq \frac{L^m}{1 - L} \|x^{(1)} - x^{(0)}\|_1 < \varepsilon$$

$$m > \frac{\log\left(\dfrac{(1 - L)\varepsilon}{\|x^{(1)} - x^{(0)}\|_1} \right)}{\log(L)}$$

$$L = \frac{2}{3}, \qquad \varepsilon = 10^{-6}, \qquad \|x^{(1)} - x^{(0)}\|_1 = \frac{2}{3}$$

$$m > \frac{\log\left(\dfrac{10^{-6}}{2} \right)}{\log\left(\dfrac{2}{3} \right)} = 35.7828 \qquad \implies \quad 36 \text{ iteradas}$$

(c)[15]

$$\begin{cases} x^{(1)} = x^{(0)} + \Delta x^{(0)}, \\ J_f(x^{(0)})\Delta x^{(0)} = -f(x^{(0)}) \end{cases}$$

$$J_f(x) = \begin{bmatrix} 1 + \frac{1}{3}\sin(x_1 - x_2) & -\frac{1}{3}\sin(x_1 - x_2) \\ \frac{1}{3}\cos(x_1 + x_2) & 1 + \frac{1}{3}\cos(x_1 + x_2) \end{bmatrix}$$

$$x^{(0)} = \begin{bmatrix} \frac{\pi}{8} & \frac{\pi}{8} \end{bmatrix}^T$$

$$J_f(x^{(0)}) = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{2}}{6} & 1 + \frac{\sqrt{2}}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.235702 & 1.23570 \end{bmatrix}$$

$$f(x^{(0)}) = \begin{bmatrix} \frac{\pi}{8} - \frac{1}{3} \\ \frac{\pi}{8} - \frac{1}{3} + \frac{\sqrt{2}}{6} \end{bmatrix} = \begin{bmatrix} 0.0593657 \\ 0.295068 \end{bmatrix}$$

$$\Delta x^{(0)} = \begin{bmatrix} \frac{1}{3} - \frac{\pi}{8} \\ \frac{16(3 - 2\sqrt{2}) + 3(-6 + \sqrt{2})\pi}{24(6 + \sqrt{2})} \end{bmatrix} = \begin{bmatrix} -0.0593657 \\ -0.227462 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} \frac{1}{3} \\ \frac{8(3 - 2\sqrt{2}) + 3\sqrt{2}\pi}{12(6 + \sqrt{2})} \end{bmatrix} = \begin{bmatrix} 0.333333 \\ 0.165237 \end{bmatrix}$$

[3][20]

Vamos mostrar que $r_\sigma\left(C_{SOR}(\omega)\right) < 1$ se e só se $\omega \in]0,2[$.

$$C_{SOR}(\omega) = I - M^{-1}(\omega)A, \qquad M(\omega) = \frac{D}{\omega} + L, \qquad A = D + L + U$$

Equação dos vectores próprios da matriz $C_{SOR}(\omega)$

$$C_{SOR}(\omega)v = \lambda v$$

$$\left(I - M^{-1}A\right)v = \lambda v$$

$$(1-\lambda)Mv = Av$$

$$\frac{1}{1-\lambda} = \frac{v^*Mv}{v^*Av}$$

$$\frac{1}{1-\bar{\lambda}} = \frac{v^*M^*v}{v^*A^*v} = \frac{v^*M^*v}{v^*Av} \quad \text{pois} \quad A^* = A$$

$$\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} = \frac{v^*(M+M^*)v}{v^*Av}$$

Sendo $A = A^*$ então $D = D^*$, $U^* = L$ pelo que:

$$M + M^* = \frac{D+D^*}{\omega} + L + L^* = A + \frac{2-\omega}{\omega}D$$

$$\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} = 1 + \frac{2-\omega}{\omega}\rho, \qquad \rho := \frac{v^*Dv}{v^*Av}$$

Sendo A definida positiva também D será definida positiva e $\rho > 0$.

Condição suficiente: sendo $\omega \in]0,2[$ tem-se

$$\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} > 1 \iff |\lambda| < 1 \qquad \Longrightarrow r_\sigma\left(C_{SOR}(\omega)\right) < 1$$

Condição necessária: sendo $\omega \in [2,\infty[$ tem-se

$$\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} \leq 1 \iff |\lambda| \geq 1 \qquad \Longrightarrow r_\sigma\left(C_{SOR}(\omega)\right) \geq 1$$

[4]

(a)[20]

Fórmula de Newton com diferenças divididas:

i	x_i	$f[x_i]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	1	-2			
			1		
1	2	-1		$\frac{1}{2}$	
			2		$-\frac{1}{6}$
2	3	1		0	
			2		
3	4	3			

$$\begin{aligned} p_3(x) &= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_4](x-x_0)(x-x_1) \\ &\quad + f[x_0,x_1,x_4,x_5](x-x_0)(x-x_1)(x-x_4) \end{aligned}$$

$$p_3(x) = -2 + (x-1) + \frac{1}{2}(x-1)(x-2) - \frac{1}{6}(x-1)(x-2)(x-3)$$

$$p_3(x) = -\frac{x^3}{6} + \frac{3x^2}{2} - \frac{7x}{3} - 1$$

(b)[15]

Método da secante:

$$x_m = g(x_{m-1}, x_{m-2}), \quad m \geq 2, \qquad g(x,y) = \frac{yf(x) - xf(y)}{f(x) - f(y)}$$

$$f(x) = p_3(x), \qquad x_0 = 2, \quad x_1 = 3$$

$$x_2 = \frac{5}{2} = 2.5 \qquad x_3 = \frac{43}{17} = 2.52941$$

(c)[15]

i	0	1	2	3
y_i	-2	-1	1	3
$g(y_i)$	1	2	3	4

$$q_3(y) = \sum_{i=0}^{3} g(y_i) l_{i,3}(y), \qquad l_{i,3}(y) = \prod_{j=0, j\neq i}^{3} \frac{y - y_j}{y_i - y_j}$$

$$z \approx q_3(0) = l_{0,3}(0) + 2l_{1,3}(0) + 3l_{2,3}(0) + 4l_{3,3}(0)$$

$$l_{0,3}(y) = \frac{(y+1)(y-1)(y-3)}{(-2+1)(-2-1)(-2-3)}$$

$$l_{1,3}(y) = \frac{(y+2)(y-1)(y-3)}{(-1+2)(-1-1)(-1-3)}$$

$$l_{2,3}(y) = \frac{(y+2)(y+1)(y-3)}{(1+2)(1+1)(1-3)}$$

$$l_{3,3}(y) = \frac{(y+2)(y+1)(y-1)}{(3+2)(3+1)(3-1)}$$

$$l_{0,3}(0) = -\frac{1}{5} \qquad l_{1,3}(0) = \frac{3}{4} \qquad l_{2,3}(0) = \frac{1}{2} \qquad l_{3,3}(0) = -\frac{1}{20}$$

$$z \approx q_3(0) = \frac{13}{5} = 2.6$$

[5]
(a)[10]

Fórmula de Gauss-Legendre de ordem 2:

$$I_2(f) = \frac{5}{9}\, f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\, f(0) + \frac{5}{9}\, f\left(\sqrt{\frac{3}{5}}\right)$$

$$I_2(f) = \frac{5}{9} \times 2.04358 + \frac{8}{9} \times 2.71828 + \frac{5}{9} \times 2.04358 = 4.6869$$

(b)[15]

Erro da fórmula de Gauss-Legendre composta de ordem 2:

$$E_2^{(M)}(f) = \frac{b-a}{2}\left(\frac{h_M}{2}\right)^6 \frac{1}{15750}\, f^{(6)}(\eta), \qquad \eta \in]a, b[$$

$$h_M = \frac{b-a}{M}$$

$$\left|E_2^{(M)}(f)\right| \leq \frac{1}{15750 M^6} \max_{x\in[-1,1]} \left|\frac{d^6}{dx^6}\, e^{\cos x}\right|$$

$$\frac{31e}{15750 M^6} < 10^{-6} \iff M > \left(\frac{31e10^6}{15750}\right)^{1/6} = 4.18211$$

$$\Longrightarrow \quad 5 \ \text{sub-intervalos}$$

[6]

$$\begin{cases} W'(x) = F(x, W(x)), & x \geq x_0 \\ W(x_0) = W_0 \end{cases}$$

$$W = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad F(x, W) = \begin{bmatrix} z \\ g(x, y, z) \end{bmatrix}, \quad x_0 = 2, \quad W_0 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$g(x, y, z) = -y - x^2 z$$

(a)[20]

Método de Euler modificado (passo h):

$$W_1 = W_0 + hF\left(\tilde{x}_0, \tilde{W}_0\right)$$

$$\tilde{x}_0 = x_0 + \frac{h}{2}, \qquad \tilde{W}_0 = W_0 + \frac{h}{2} F(x_0, W_0)$$

$$\tilde{W}_0 = \begin{bmatrix} \tilde{y}_0 \\ \tilde{z}_0 \end{bmatrix} = \begin{bmatrix} y_0 + \frac{h}{2} z_0 \\ z_0 + \frac{h}{2} g(x_0, y_0, z_0) \end{bmatrix}$$

$$W_1 = \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} y_0 + h\tilde{z}_0 \\ z_0 + hg(\tilde{x}_0, \tilde{y}_0, \tilde{z}_0) \end{bmatrix}$$

$$x_0 = 2, \qquad y_0 = 4, \qquad z_0 = 3$$

$$g(x_0, y_0, z_0) = -16$$

$$\begin{bmatrix} \tilde{y}_0 \\ \tilde{z}_0 \end{bmatrix} = \begin{bmatrix} 4 + \dfrac{3h}{2} \\ 3 - 8h \end{bmatrix}$$

$$g(\tilde{x}_0, \tilde{y}_0, \tilde{z}_0) = -16 + \frac{49h}{2} + \frac{61h^2}{4} + 2h^3$$

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 4 + 3h - 8h^2 \\ 3 - 16h + \frac{49h^2}{2} + \frac{61h^3}{4} + 2h^4 \end{bmatrix}$$

(b)[20]

Método de Adams-Moulton de ordem 2 (passo h):

$$W_1 = W_0 + \frac{h}{2}\left[F(x_0, W_0) + F(x_1, W_1)\right]$$

$$x_1 = x_0 + h$$

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} + \frac{h}{2}\begin{bmatrix} z_0 + z_1 \\ -y_0 - x_0^2 z_0 - y_1 - x_1^2 z_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{h}{2} \\ \frac{h}{2} & 1 + \frac{h}{2}\,x_1^2 \end{bmatrix}\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} y_0 + \frac{h}{2}\,z_0 \\ z_0 - \frac{h}{2}(y_0 + x_0^2 z_0) \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \frac{1}{1 + \frac{h}{2}\,x_1^2 + \frac{h^2}{4}}\begin{bmatrix} 1 + \frac{h}{2}\,x_1^2 & \frac{h}{2} \\ -\frac{h}{2} & 1 \end{bmatrix}\begin{bmatrix} y_0 + \frac{h}{2}\,z_0 \\ z_0 - \frac{h}{2}(y_0 + x_0^2 z_0) \end{bmatrix}$$

$$x_0 = 2, \qquad y_0 = 4, \qquad z_0 = 3$$

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \frac{1}{1 + 2h + \frac{9h^2}{4} + \frac{h^3}{2}}\begin{bmatrix} 1 + 2h + 2h^2 + \frac{h^3}{2} & \frac{h}{2} \\ -\frac{h}{2} & 1 \end{bmatrix}\begin{bmatrix} 4 + \frac{3h}{2} \\ 3 - 8h \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \frac{1}{4 + 8h + 9h^2 + 2h^3}\begin{bmatrix} 16 + 44h + 28h^2 + 20h^3 + 3h^4 \\ 12 - 40h - 3h^2 \end{bmatrix}$$