

INSTITUTO SUPERIOR TÉCNICO
Mestrado em Engenharia Física Tecnológica

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Semestre: 1^o

MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 28.JAN.2010

$[1]^{20}$

$$\begin{cases} z_1 = a_2 \times x \\ z_2 = z_1 + a_1 \\ z_3 = z_2 \times x \\ z = z_3 + a_0 \end{cases}$$

$$\delta_{\tilde{z}_1} = \delta_{\tilde{a}_2} + \delta_{\tilde{x}} + \delta_1 = \delta_1$$

$$\delta_{\tilde{z}_2} = \frac{z_1}{z_2} \delta_{\tilde{z}_1} + \frac{a_1}{z_2} \delta_{\tilde{a}_1} + \delta_2 = \frac{z_1}{z_2} \delta_1 + \delta_2$$

$$\delta_{\tilde{z}_3} = \delta_{\tilde{z}_2} + \delta_{\tilde{x}} + \delta_3 = \frac{z_1}{z_2} \delta_1 + \delta_2 + \delta_3$$

$$\delta_{\tilde{z}} = \frac{z_3}{z} \delta_{\tilde{z}_3} + \frac{a_0}{z} \delta_{\tilde{a}_0} + \delta_4 = \frac{z_3}{z} \left(\frac{z_1}{z_2} \delta_1 + \delta_2 + \delta_3 \right) + \delta_4$$

$$|\delta_i| \leq U, \quad i = 1, 2, 3, 4$$

Erros de arredondamento das operações aritméticas

$$\tilde{z} = (1 - \delta_{\tilde{z}})z$$

$$= (1 - \delta_4)z - (\delta_2 + \delta_3)z_3 - \frac{z_3 z_1}{z_2} \delta_1$$

$$= (1 - \delta_4)(a_0 + a_1 x + a_2 x^2) - (\delta_2 + \delta_3)(a_1 x + a_2 x^2) - \delta_1 a_2 x^2$$

$$= (1 - \delta_4)a_0 + (1 - \delta_2 - \delta_3 - \delta_4)a_1 x + (1 - \delta_1 - \delta_2 - \delta_3 - \delta_4)a_2 x^2$$

$$A_0 = (1 - \delta_4)a_0, \quad A_1 = (1 - \delta_2 - \delta_3 - \delta_4)a_1, \quad A_2 = (1 - \delta_1 - \delta_2 - \delta_3 - \delta_4)a_2$$

$$\delta_{A_i} = 1 - \frac{A_i}{a_i}, \quad i = 0, 1, 2$$

$$|\delta_{A_0}| = |\delta_4| \leq U, \quad |\delta_{A_1}| = |\delta_2 + \delta_3 + \delta_4| \leq 3U, \quad |\delta_{A_2}| = |\delta_1 + \delta_2 + \delta_3 + \delta_4| \leq 4U$$

[2]
(a)²⁰

Condições suficientes de convergência do método da secante:

$$(0) \ p \in C^2(I)$$

$$(i) \ p(0.2) = 0.800064, \quad p(0.4) = -0.395904, \quad p(0.2)p(0.4) < 0$$

$$(ii) \ p'(x) = 6x^5 - 6 < 0, \quad \forall x \in I$$

$$(iii) \ p''(x) = 30x^4 > 0, \quad \forall x \in I$$

$$(iv) \ \left| \frac{p(0.2)}{p'(0.2)} \right| = 0.133387 < 0.2, \quad \left| \frac{p(0.4)}{f'(0.4)} \right| = 0.0666667 < 0.2$$

(b)²⁰

$$x_{m+1} = x_m - p(x_m) \frac{x_m - x_{m-1}}{p(x_m) - p(x_{m-1})}, \quad m \geq 0$$

$$|z - x_m| \leq B_m, \quad B_m = K |z - x_{m-1}| |z - x_{m-2}|, \quad m \geq 2$$

$$K = \frac{\max_{x \in I} |p''(x)|}{2 \min_{x \in I} |p'(x)|} = \frac{|p''(0.4)|}{2|p'(0.4)|} = 0.0646621$$

m	x_m	B_m
0	0.2	0.2
1	0.4	0.2
2	0.333794	0.258649×10^{-2}
3	0.333562	0.334495×10^{-4}

$$z = 0.333562 + \varepsilon, \quad |\varepsilon| \leq 0.334495 \times 10^{-4}.$$

[3]²⁰

Método de Newton generalizado:

$$\begin{cases} x^{(1)} = x^{(0)} + \Delta x^{(0)}, \\ J_f(x^{(0)}) \Delta x^{(0)} = -f(x^{(0)}) \end{cases}$$

$$J_f(x) = \begin{bmatrix} 8x_1 & -2x_2 \\ -\cos x_1 & 2 + \sin x_2 \end{bmatrix}$$

$$x^{(0)} = \begin{bmatrix} 0.25 \\ 0.50 \end{bmatrix}$$

$$\begin{bmatrix} 2.0 & -1.0 \\ -0.968912 & 2.47943 \end{bmatrix} \Delta x^{(0)} = \begin{bmatrix} 0.0 \\ 0.124987 \end{bmatrix}$$

$$\begin{bmatrix} 2.0 & -1.0 \\ 0.0 & 1.99497 \end{bmatrix} \Delta x^{(0)} = \begin{bmatrix} 0.0 \\ 0.124987 \end{bmatrix}$$

$$\Delta x^{(0)} = \begin{bmatrix} 0.0313255 \\ 0.0626509 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 0.281326 \\ 0.562651 \end{bmatrix}$$

[4]²⁰

Se $x = x_j$, para algum $j = 0, \dots, n$, o resultado é trivialmente verdadeiro.

Seja $x \in [a, b] \setminus \{x_0, \dots, x_n\}$ e consideremos a função auxiliar $F : [a, b] \rightarrow \mathbb{R}$ definida por:

$$F(t) = e_n(t)W_{n+1}(x) - e_n(x)W_{n+1}(t), \quad W_{n+1}(x) = \prod_{i=0}^n (x - x_i)$$

F tem pelo menos $n + 2$ zeros distintos em $[a, b]$ uma vez que

$$F(x) = 0, \quad F'(x_j) = 0, \quad j = 0, \dots, n$$

F' tem pelo menos $n + 1$ zeros em $]x_0; x_1; \dots; x_n; x[$ (pelo teorema de Rolle).

$F^{(n+1)}$ tem pelo menos um zero nesse intervalo (por indução), que designamos por ξ .

Derivando F $n + 1$ vezes resulta:

$$\begin{aligned} F^{(n+1)}(t) &= e_n^{(n+1)}(t)W_{n+1}(x) - e_n(x)W_{n+1}^{(n+1)}(t) \\ &= f^{(n+1)}(t)W_{n+1}(x) - e_n(x)(n+1)! \end{aligned}$$

Fazendo $t = \xi$, resulta

$$0 = F^{(n+1)}(\xi) = f^{(n+1)}(\xi)W_{n+1}(x) - e_n(x)(n+1)!,$$

donde se obtém o resultado pretendido:

$$e_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} W_{n+1}(x)$$

[5]
(a)²⁰

Polinómios de Chebyshev obtidos pela fórmula de recorrência:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$

$$T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x$$

Representação das potências de x em termos dos polinómios de Chebyshev:

$$1 = T_0(x)$$

$$x = T_1(x)$$

$$x^2 = \frac{1}{2}[T_0(x) + T_2(x)]$$

$$x^3 = \frac{1}{4}[3T_1(x) + T_3(x)]$$

Representação de F em termos dos polinómios de Chebyshev:

$$F(x) = \frac{5}{4}T_0(x) + \frac{9}{8}T_1(x) + \frac{1}{4}T_2(x) + \frac{1}{24}T_3(x)$$

Melhor aproximação mínimos quadrados (base ortogonal):

$$p_2^*(x) = a_0^*T_0(x) + a_1^*T_1(x) + a_2^*T_2(x)$$

$$a_k^* = \frac{\langle F, T_k \rangle}{\langle T_k, T_k \rangle}, \quad k = 0, 1, 2$$

$$a_0^* = \frac{5}{4}, \quad a_1^* = \frac{9}{8}, \quad a_2^* = \frac{1}{4}$$

$$p_2^*(x) = 1 + \frac{9}{8}x + \frac{1}{2}x^2$$

(b)²⁰

$$x = X(t) = -1 + 2(t + 1) = 2t + 1, \quad t \in [-1, 1]$$

$$(x + 1)(3 - x) = 4(1 - t^2)$$

$$I(f) = \int_{-1}^1 \frac{\tilde{f}(t)}{\sqrt{1 - t^2}} dt = I(\tilde{f}, [-1, 1]), \quad \tilde{f}(t) = f(X(t))$$

$$I(\tilde{f}, [-1, 1]) \approx I_2(\tilde{f}) = w_{0,2}\tilde{f}(x_{0,2}) + w_{1,2}\tilde{f}(x_{1,2}) + w_{2,2}\tilde{f}(x_{2,2})$$

$$w_{j,2} = \frac{\pi}{3}, \quad j = 0, 1, 2$$

$$x_{j,2} = \cos\left(\frac{2j+1}{6}\pi\right), \quad j = 0, 1, 2$$

$$x_{0,2} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad x_{1,2} = \cos\left(\frac{\pi}{2}\right) = 0, \quad x_{2,2} = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$I_2(\tilde{f}) = \frac{\pi}{3} \left[\tilde{f}\left(-\frac{\sqrt{3}}{2}\right) + \tilde{f}(0) + \tilde{f}\left(\frac{\sqrt{3}}{2}\right) \right]$$

$$I(f) \approx I_2(\tilde{f}) = \frac{\pi}{3} \left[f(1 - \sqrt{3}) + f(1) + f(1 + \sqrt{3}) \right]$$

[6]²⁰

Erro de integração da regra do ponto médio composta:

$$E_0^{(M)}(f) = \frac{(b-a)^3}{6M^2} f''(\xi)$$

$$\left| E_0^{(M)}(f) \right| \leq \frac{(b-a)^3}{6M^2} B_2 < \varepsilon, \quad B_2 = \max_{x \in [a,b]} |f''(x)|$$

$$M > \sqrt{\frac{(b-a)^3 B_2}{6\varepsilon}}$$

$$f(x) = e^{-x^2}, \quad a = 1, \quad b = 2, \quad \varepsilon = 10^{-6}$$

$$f'(x) = -2xf(x), \quad f''(x) = (4x^2 - 2)f(x)$$

$$f'''(x) = (-8x^3 + 12x)f(x)$$

$$f'''(x) = 0 \quad \Leftrightarrow \quad x = 0 \cup x = \sqrt{\frac{3}{2}} \cup x = -\sqrt{\frac{3}{2}}$$

$$\begin{aligned} B_2 &= \max \left\{ |f''(1)|, |f''(2)|, \left| f''\left(\sqrt{\frac{3}{2}}\right) \right| \right\} = \max \{ 2e^{-1}, 14e^{-4}, 4e^{-3/2} \} \\ &= \max \{ 0.735759, 0.256419, 0.892521 \} = 0.892521 \end{aligned}$$

$$M > 385.686, \quad M = 386$$

[7] $f(x, y) = y^3 + 2xy + 3$

(a)¹⁰ Método de Taylor de 2ª ordem

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2} (d_f f)(x_0, y_0)$$

$$(d_f f)(x, y) = \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) (x, y) = 2y + f(x, y)(3y^2 + 2x)$$

$$x_0 = 1, \quad y_0 = 1$$

$$f(x_0, y_0) = 6, \quad (d_f f)(x_0, y_0) = 32$$

$$y_1 = 1 + 6h + 16h^2$$

(b)¹⁵ Método de Adams-Bashforth de 2ª ordem (passo h):

$$\hat{y}_2 = \hat{y}_1 + \frac{h}{2} [3f(x_1, \hat{y}_1) - f(x_0, \hat{y}_0)]$$

$$x_0 = 1, \quad \hat{y}_0 = 1, \quad x_1 = 1 + h, \quad \hat{y}_1 = 1 + 6h + 16h^2$$

$$f(x_0, \hat{y}_0) = 6$$

$$f(x_1, \hat{y}_1) = 3 + 2(1 + h)(1 + 6h + 16h^2) + (1 + 6h + 16h^2)^3$$

$$= 6 + 32h + 200h^2 + 824h^3 + 2496h^4 + 4608h^5 + 4096h^6$$

$$\hat{y}_2 = 1 + 12h + 64h^2 + 300h^3 + 1236h^4 + 3744h^5 + 6912h^6 + 6144h^7$$

[8]¹⁵

$$W'(x) = F(x, W(x)), \quad W = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}, \quad F(x, W) = \begin{bmatrix} z \\ \frac{y}{x} + z \end{bmatrix}$$

Método de Heun (passo h):

$$W_1 = W_0 + \frac{h}{2} \left\{ F(x_0, W_0) + F(\tilde{x}_1, \tilde{W}_1) \right\}$$

$$\tilde{x}_1 = x_0 + h = 1 + h, \quad \tilde{W}_1 = W_0 + hF(x_0, W_0)$$

$$W_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad F(x_0, W_0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\tilde{W}_1 = \begin{bmatrix} \tilde{y}_1 \\ \tilde{z}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + h \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 + 2h \\ 2 + 3h \end{bmatrix}$$

$$F\left(\tilde{x}_1, \tilde{W}_1\right)=\left[\begin{array}{c} \tilde{z}_0 \\ \frac{\tilde{y}_0}{\tilde{x}_0}+\tilde{z}_0 \end{array}\right]=\left[\begin{array}{c} 2+3 h \\ \frac{1+2 h}{1+h}+2+3 h \end{array}\right]$$

$$W_1=\left[\begin{array}{c} y_1 \\ z_1 \end{array}\right]=\left[\begin{array}{c} 1+\frac{h}{2}(4+3 h) \\ 2+\frac{h}{2}\left(5+3 h+\frac{1+2 h}{1+h}\right) \end{array}\right]$$