

INSTITUTO SUPERIOR TÉCNICO
Mestrado em Engenharia Física Tecnológica
Ano Lectivo: 2009/2010

MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 16.JAN.2010

[1]²⁰

$$z_1 = f(a), \quad z_2 = f'(a), \quad z_3 = \frac{z_1}{z_2}, \quad z = a - z_3$$

$$\delta_{z_1} = p_f(a)\delta_{\bar{a}} + \delta_f, \quad p_f(a) = \frac{af'(a)}{f(a)}$$

$$\delta_{z_2} = p_{f'}(a)\delta_{\bar{a}} + \delta_{f'}, \quad p_{f'}(a) = \frac{af''(a)}{f'(a)}$$

$$\delta_{z_3} = \delta_{z_1} - \delta_{z_2} + \delta_d = [p_f(a) - p_{f'}(a)]\delta_{\bar{a}} + \delta_f - \delta_{f'} + \delta_d$$

$$\delta_{\bar{z}} = \frac{a}{z} \delta_{\bar{a}} - \frac{z_3}{z} \delta_{z_3} + \delta_s$$

$$\delta_{\bar{z}} = \frac{1}{z} \{a - z_3[p_f(a) - p_{f'}(a)]\} \delta_{\bar{a}} - \frac{z_3}{z} (\delta_f - \delta_{f'} + \delta_d) + \delta_s$$

$$\delta_{\bar{z}} = \frac{a}{z} \frac{f(a)f''(a)}{[f'(a)]^2} \delta_{\bar{a}} - \frac{1}{z} \frac{f(a)}{f'(a)} (\delta_f - \delta_{f'} + \delta_d) + \delta_s$$

[2]

(a)¹⁵

$$p(x) = 0 \quad \Longleftrightarrow \quad x = g(x) \quad (x \neq 0)$$

$$g(x) = \frac{1}{8}(x^4 - 4), \quad g'(x) = \frac{x^3}{2}, \quad g''(x) = \frac{3x^2}{2}$$

Condições suficientes de convergência do método do ponto fixo para z_1 ,
 $\forall x_0 \in I_1 = [-0.6, -0.4]$:

(i) $g \in C^1(I_1)$

(ii) $\max_{x \in I_1} |g'(x)| = |g'(-0.6)| = 0.108 < 1$

(ii) $g(I_1) \subset I_1$, pois $g(-0.6) = -0.4838 \in I_1$,

$$g(-0.4) = -0.4968 \in I_1, \quad g'(x) < 0, \quad \forall x \in I_1$$

(b)²⁰Método do ponto fixo: $x_m = g(x_{m-1})$, $m \geq 1$ Critério de paragem: $|z - x_m| \leq K|x_m - x_{m-1}|$

$$L = \max_{x \in I_1} |g'(x)| = 0.108, \quad K = \frac{L}{1-L} \approx 0.121076$$

m	x_m	$K x_m - x_{m-1} $
0	-0.5	
1	-0.492188	0.946×10^{-3}
2	-0.492664	0.578×10^{-4}

$$z_1 = -0.492664 + \varepsilon, \quad |\varepsilon| \leq 0.578 \times 10^{-4}$$

[3]

(a)¹⁰

Uma condição suficiente de convergência é que a matriz A tenha diagonal estritamente dominante por linhas, isto é, que

$$|a_{11}| > |a_{12}| + |a_{13}|, \quad |a_{22}| > |a_{21}| + |a_{23}|, \quad |a_{33}| > |a_{31}| + |a_{32}|.$$

Esta condição é satisfeita para $\alpha \in]-1, 1[$.

(b)²⁵

$$A = \begin{bmatrix} 2 & \frac{1}{2} & 0 \\ 1 & 2 & \frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix} = D + L + U = M_J + N_J$$

$$M_J = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad N_J = L + U = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$x^{(m+1)} = D^{-1} (b - (L + U)x^{(m)}), \quad m \geq 0$$

$$x^{(m+1)} = \frac{1}{2} (-N_J x^{(m)} + b), \quad m \geq 0$$

$$x^{(0)} = 0$$

$$x^{(1)} = \frac{1}{2} b = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 3 \end{bmatrix}$$

$$x^{(2)} = \frac{1}{2} (-N_J x^{(1)} + b) = \frac{1}{2} \left(\begin{bmatrix} -\frac{7}{4} \\ -4 \\ -\frac{7}{2} \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} \frac{13}{8} \\ \frac{3}{2} \\ \frac{5}{4} \end{bmatrix}$$

$$\|z - x^{(2)}\|_\infty \leq \frac{c}{1-c} \|x^{(2)} - x^{(1)}\|_\infty, \quad c = \|C_J\|_\infty$$

$$C_J = -M_J^{-1}N_J = -\frac{1}{2}N_J$$

$$\|C_J\|_\infty = \max \left\{ \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \right\} = \frac{3}{4}$$

$$x^{(2)} - x^{(1)} = \begin{bmatrix} -\frac{7}{8} & -2 & -\frac{7}{4} \end{bmatrix}^T$$

$$\|x^{(2)} - x^{(1)}\|_\infty = \max \left\{ \frac{7}{8}, 2, \frac{7}{4} \right\} = 2$$

$$\|z - x^{(2)}\|_\infty \leq 6$$

[4]

(a)¹⁵

Fórmula de Newton às diferenças divididas:

i	x_i	$f[x_i]$	$f[\cdot, \cdot]$	$f[\cdot, \cdot, \cdot]$
0	-2	-1		
			3	
1	0	5		2
			11	
2	2	27		

$$\begin{aligned} p_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= -1 + 3(x + 2) + 2(x + 2)x \\ &= 5 + 7x + 2x^2 \end{aligned}$$

(b)¹⁵

$$e_2(x) = f(x) - p_2(x) = \frac{f'''(\xi)}{3!} W_3(x), \quad x \in [-2, 2]$$

$$W_3(x) = (x + 2)x(x - 2) = x(x^2 - 4), \quad \xi \in \text{int}[-2; 2; x] \subset [-2, 2]$$

$$|e_2(x)| \leq \frac{1}{6} \max_{x \in [-2, -2]} |f'''(x)| \max_{x \in [-2, 2]} |W_3(x)|, \quad \forall x \in [-2, 2]$$

$$\max_{x \in [-2, -2]} |f'''(x)| \leq 1$$

$$W_3'(x) = 3x^2 - 4 = 3(x+a)(x-a), \quad a = \frac{2}{\sqrt{3}}$$

$$\max_{x \in [-2, 2]} |W_3(x)| = |W_3(a)| = \frac{16}{3\sqrt{3}} = 3.07920$$

$$|e_2(x)| \leq \frac{8}{9\sqrt{3}} = 0.513200 \quad \forall x \in [-2, 2]$$

(c)¹⁵

Pretende-se determinar a m.a.m.q.d. de f definida pelos três pontos da tabela por um polinómio do 2º grau, q , problema que tem uma solução única. Ora, para o polinómio interpolador p_2 determinado na alínea (a) tem-se $S(2, 7, 5) = 0$, o que é o menor valor possível de S .

Conclui-se que $q = p_2$ e portanto $a = 2, b = 7, c = 5$.

ALTERNATIVA: Cálculo da m.a.m.q.d. de f por um polinómio de grau 2 usando o sistema normal

$$q(x) = c\phi_0(x) + b\phi_1(x) + a\phi_2(x), \quad \phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = x^2$$

$$\begin{bmatrix} \langle \bar{\phi}_0, \bar{\phi}_0 \rangle & \langle \bar{\phi}_0, \bar{\phi}_1 \rangle & \langle \bar{\phi}_0, \bar{\phi}_2 \rangle \\ \langle \bar{\phi}_1, \bar{\phi}_0 \rangle & \langle \bar{\phi}_1, \bar{\phi}_1 \rangle & \langle \bar{\phi}_1, \bar{\phi}_2 \rangle \\ \langle \bar{\phi}_2, \bar{\phi}_0 \rangle & \langle \bar{\phi}_2, \bar{\phi}_1 \rangle & \langle \bar{\phi}_2, \bar{\phi}_2 \rangle \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} \langle \bar{f}, \bar{\phi}_0 \rangle \\ \langle \bar{f}, \bar{\phi}_1 \rangle \\ \langle \bar{f}, \bar{\phi}_2 \rangle \end{bmatrix}$$

$$\bar{\phi}_0 = [1 \ 1 \ 1]^T, \quad \bar{\phi}_1 = [-2 \ 0 \ 2]^T, \quad \bar{\phi}_2 = [4 \ 0 \ 4]^T, \quad \bar{f} = [-1 \ 5 \ 27]^T$$

$$\langle \bar{\phi}, \bar{\psi} \rangle = \sum_{i=0}^2 \bar{\phi}_i \bar{\psi}_i$$

$$\langle \bar{\phi}_0, \bar{\phi}_0 \rangle = 1 + 1 + 1 = 3$$

$$\langle \bar{\phi}_0, \bar{\phi}_1 \rangle = -2 + 0 + 2 = 0, \quad \langle \bar{\phi}_1, \bar{\phi}_0 \rangle = 0$$

$$\langle \bar{\phi}_0, \bar{\phi}_2 \rangle = 4 + 0 + 4 = 8, \quad \langle \bar{\phi}_2, \bar{\phi}_0 \rangle = 8$$

$$\langle \bar{\phi}_1, \bar{\phi}_1 \rangle = 4 + 0 + 4 = 8$$

$$\langle \bar{\phi}_1, \bar{\phi}_2 \rangle = -8 + 0 + 8 = 0, \quad \langle \bar{\phi}_2, \bar{\phi}_1 \rangle = 0$$

$$\langle \bar{\phi}_2, \bar{\phi}_2 \rangle = 16 + 0 + 16 = 32$$

$$\langle \bar{f}, \bar{\phi}_0 \rangle = -1 + 5 + 27 = 31$$

$$\langle \bar{f}, \bar{\phi}_1 \rangle = 2 + 0 + 54 = 56$$

$$\langle \bar{f}, \bar{\phi}_2 \rangle = -4 + 0 + 108 = 104$$

$$\begin{bmatrix} 3 & 0 & 8 \\ 0 & 8 & 0 \\ 8 & 0 & 32 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 31 \\ 56 \\ 104 \end{bmatrix}$$

$$c = 5, \quad b = 7, \quad a = 2$$

[5]
(a)²⁵

$$Y(1.2) = e^{-I(g)}, \quad I(g) = \int_{1.0}^{1.2} g(x) dx, \quad g(x) = \cos(x^2)$$

$$I_1^{(2)}(g) = \frac{1.2 - 1.0}{4} [g(1.0) + 2g(1.1) + g(1.2)] = 0.0688382$$

$$y_a = e^{-I_1^{(2)}(g)} = 0.933478$$

$$Y(1.2) - y_a = e^{-I_1^{(2)}(g) - E_1^{(2)}(g)} - y_a = y_a \left[e^{-E_1^{(2)}(g)} - 1 \right]$$

$$|Y(1.2) - y_a| = y_a \left| e^{-E_1^{(2)}(g)} - 1 \right|$$

$$\left| E_1^{(2)}(g) \right| \leq \frac{1}{6} \left(\frac{1.2 - 1.0}{2} \right)^3 \max_{x \in [1.0, 1.2]} |g''(x)|$$

$$g'(x) = -2x \sin(x^2), \quad g''(x) = -2 \sin(x^2) - 4x^2 \cos(x^2)$$

$$|g''(x)| \leq 2 \sin(1.44) + 4 \times 1.44 \cos(1.0) = 5.09506, \quad \forall x \in [1.0, 1.2]$$

$$\left| E_1^{(2)}(g) \right| \leq 0.849177 \times 10^{-3} \equiv \varepsilon$$

$$|Y(1.2) - y_a| \leq y_a (e^\varepsilon - 1) = 0.793025 \times 10^{-3}$$

(b)²⁵ Método de Euler (2 iteradas)

$$f(x, y) = -y \cos(x^2), \quad x_0 = 1.0, \quad y_0 = 1.0, \quad h = 0.1$$

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n \geq 0$$

$$y_1 = y_0 + hf(x_0, y_0) = 0.945970$$

$$y_b = y_2 = y_1 + hf(x_1, y_1) = 0.912575$$

$$|Y(1.2) - y_b| \leq \frac{\tau(h)}{K} (e^{0.2K} - 1)$$

$$K = \max_{x \in [1.0, 1.2]} \left| \frac{\partial f(x, y)}{\partial y} \right| = \max_{x \in [1.0, 1.2]} |\cos(x^2)| = \cos(1.0) = 0.540302$$

$$\tau(h) = \max_{0 \leq n \leq 1} |\tau(x_n, Y(x_n); h)|$$

$$\tau(x, Y(x), h) = \frac{Y(x+h) - Y(x)}{h} - f(x, Y(x)) = \frac{h}{2} Y''(x + \theta h)$$

$$\theta \in [0, 1]$$

$$Y''(x) = Y(x) [\cos^2(x^2) + 2x \sin(x^2)]$$

$$|Y''(x)| \leq \cos^2(1.0) + 2 \times 1.2 \sin(1.44) = 2.67143, \quad \forall x \in [1.0, 1.2]$$

$$\tau(h) = \frac{h}{2} \max_{x \in [1.0, 1.2]} |Y''(x)| \leq 0.05 \times 2.67143 = 0.133572$$

$$|Y(1.2) - y_b| \leq 0.0282112$$

(c)¹⁵ Método de Taylor de 2ª ordem

$$f(x, y) = -y \cos(x^2), \quad x_0 = 1, \quad y_0 = 1, \quad h = 0.2$$

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2} (d_f f)(x_0, y_0)$$

$$(d_f f)(x, y) = \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) (x, y) = [2x \sin(x^2) + \cos^2(x^2)] y$$

$$f(x_0, y_0) = -\cos(1.0) = -0.540302$$

$$(d_f f)(x_0, y_0) = 2 \sin(1.0) + \cos^2(1.0) = 1.97487$$

$$y_c = y_1 = 0.931437$$