INSTITUTO SUPERIOR TÉCNICO

Mestrado em Engenharia Electrotécnica e de Computadores

Ano Lectivo: 2007/2008 Semestre: 2º

MATEMÁTICA COMPUTACIONAL

Formulário – II

6. Interpolação Polinomial

Fórmula interpoladora de Lagrange:

$$p_n(x) = \sum_{j=0}^{n} f_j \, l_j(x), \qquad l_j(x) = \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}$$

Fórmula interpoladora de Newton:

$$p_n(x) = f[x_0] + \sum_{j=1}^{n} f[x_0, x_1, \ldots, x_j] \, W_j(x)$$

$$W_j(x) = \prod_{i=0}^{j-1} (x - x_i), \qquad j = 1, 2, \ldots, n$$

$$f[x_0] = f(x_0), \qquad f[x_0, x_1, \ldots, x_j] = \sum_{l=0}^{j} \frac{f(x_l)}{\displaystyle\prod_{i=0, i\neq l}^{j} (x_l - x_i)}, \qquad j = 1, 2, \ldots, n$$

$$f[x_0, x_1, \ldots, x_j] = \frac{f[x_1, x_2, \ldots, x_j] - f[x_0, x_1, \ldots, x_{j-1}]}{x_j - x_0}, \qquad j = 1, 2, \ldots, n$$

Fórmula do erro:

$$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} W_{n+1}(x) = f[x_0, x_1, \ldots, x_n, x] \, W_{n+1}(x)$$

$$W_{n+1}(x) = \prod_{i=0}^{n} (x - x_i), \qquad \xi \in]x_0; x_1; \ldots; x_n; x[$$

Relação entre as diferenças divididas e as derivadas de uma função:

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}, \qquad \xi \in]x_0; x_1; \ldots; x_n[$$

7. Aproximação Mínimos Quadrados

Melhor aproximação mínimos quadrados ϕ^* de $f \in E$ em $F \subset E$, F subespaço de dimensão n gerado por $\{\varphi_0, \ldots, \varphi_n\}$, E espaço pré-Hilbertiano:

$$\|f - \phi^*\|_2 = \min_{\phi \in F} \|f - \phi\|_2 \quad \Leftrightarrow \quad \langle f - \phi^*, \phi \rangle = 0, \quad \forall \phi \in F$$

$$\phi^* = \sum_{k=0}^{n} a_k^* \varphi_k, \qquad \sum_{k=0}^{n} \langle \varphi_j, \varphi_k \rangle a_k^* = \langle f, \varphi_j \rangle, \quad j = 0, 1, \ldots, n$$

$$\phi^* = \sum_{k=0}^{n} a_k^* \varphi_k, \quad a_k^* = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}, \qquad \text{se } \{\varphi_0, \ldots, \varphi_n\} \text{ é um sistema ortogonal}$$

Polinómios ortogonais com respeito ao produto interno

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx, \quad (f, g \in C([a, b]),\ w \in C(]a, b[),\ w(x) \geq 0)$$

- Fórmula de recorrência:

$$\begin{cases} \varphi_{n+1}(x) = (x - B_{n+1})\varphi_n(x) - C_{n+1}\varphi_{n-1}(x), & n = 1, 2, \ldots \\ \varphi_0(x) = 1, \qquad \varphi_1(x) = x - B_1 \end{cases}$$

$$B_{n+1} = \frac{\langle x\varphi_n, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}, \quad n = 0, 1, \ldots, \qquad C_{n+1} = \frac{\langle x\varphi_n, \varphi_{n-1} \rangle}{\langle \varphi_{n-1}, \varphi_{n-1} \rangle}, \quad n = 1, 2, \ldots$$

- Polinómios de Legendre, P_n $\quad (x \in [a, b] = [-1, 1],\ w(x) = 1)$:

$$\begin{cases} P_{n+1}(x) = \dfrac{2n+1}{n+1}\, x P_n(x) - \dfrac{n}{n+1}\, P_{n-1}(x), & n = 1, 2, \ldots \\ P_0(x) = 1, \qquad P_1(x) = x \end{cases}$$

$$\langle P_n, P_m \rangle = 0, \quad \forall n \neq m, \qquad \langle P_n, P_n \rangle = \frac{2}{2n+1}, \quad n = 0, 1, \ldots$$

$$A_n = \lim_{x \to \infty} x^{-n} P_n(x) = \frac{(2n)!}{2^n (n!)^2}, \qquad n = 1, 2, \ldots$$

- Polinómios de Chebyshev, T_n $\quad \left(x \in [a, b] = [-1, 1],\ w(x) = 1/\sqrt{1 - x^2}\right)$:

$$\begin{cases} T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), & n = 1, 2, \ldots \\ T_0(x) = 1, \qquad T_1(x) = x \end{cases}$$

$$\langle T_n, T_m \rangle = 0, \quad \forall n \neq m, \qquad \langle T_0, T_0 \rangle = \pi, \qquad \langle T_n, T_n \rangle = \frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$A_n = \lim_{x \to \infty} x^{-n} T_n(x) = 2^{n-1}, \qquad n = 1, 2, \ldots$$

$$T_n(x) = \cos(n \arccos x), \qquad n = 0, 1, \ldots$$

$$T_n(x_i) = 0, \qquad x_i = -\cos \frac{(2i+1)\pi}{2n}, \qquad i = 0, \ldots, n-1, \qquad n = 1, 2, \ldots$$

8. Integração Numérica

Fórmulas de Newton-Cotes fechadas de ordem n ($f \in C([a,b])$):

$$I(f) = \int_a^b f(x)\,dx \qquad \approx \qquad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0, 1, \ldots, n$$

$$x_{j,n} = a + jh, \quad j = 0, 1, \ldots, n, \qquad h = \frac{b-a}{n}$$

$$w_{j,n} = I(l_{j,n}) = \frac{h(-1)^{n-j}}{j!(n-j)!} \int_0^n \prod_{i=0, i\neq j}^{n} (t-i)\,dt, \qquad w_{j,n} = w_{n-j,n}$$

$$E_n(f) = I(f) - I_n(f) = C_n h^{n+1+\nu_n} f^{(n+\nu_n)}(\xi)$$

$$C_n = \frac{1}{(n+\nu_n)!} \int_0^n t^{\nu_n - 1} \prod_{i=0}^{n} (t-i)\,dt, \qquad \nu_n = 1 + \frac{1}{2}\left[1 + (-1)^n\right], \qquad \xi \in]a,b[$$

- $n = 1,\ h = b - a$ (Regra dos trapézios):

$$I_1(f) = \frac{b-a}{2}[f(a) + f(b)], \qquad E_1(f) = -\frac{h^3}{12} f''(\xi), \quad \xi \in]a,b[$$

- $n = 2,\ h = \dfrac{b-a}{2}$ (Regra de Simpson):

$$I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right], \qquad E_2(f) = -\frac{h^5}{90} f^{(4)}(\xi)$$

- $n = 3,\ h = \dfrac{b-a}{3}$ (Regra dos três oitavos):

$$I_3(f) = \frac{b-a}{8}\left[f(a) + 3f(a+h) + 3f(b-h) + f(b)\right], \quad E_3(f) = -\frac{3h^5}{80} f^{(4)}(\xi)$$

- $n = 4,\ h = \dfrac{b-a}{4}$ (Regra de Milne):

$$I_4(f) = \frac{b-a}{90}\left[7f(a) + 32f(a+h) + 12f\left(\frac{a+b}{2}\right) + 32f(b-h) + 7f(b)\right]$$

$$E_4(f) = -\frac{8h^7}{945} f^{(6)}(\xi), \qquad \xi \in]a,b[$$

- $n = 5,\ h = \dfrac{b-a}{5}$:

$$I_5(f) = \frac{b-a}{288}\left[19f(a) + 75f(a+h) + 50f(a+2h) + 50f(b-2h) + 75f(b-h) + 19f(b)\right]$$

$$E_5(f) = -\frac{275h^7}{12096} f^{(6)}(\xi), \qquad \xi \in]a,b[$$

- $n = 6,\ h = \dfrac{b-a}{6}$ (Regra de Weddle):

$$I_6(f) = \frac{b-a}{840}\left[41f(a) + 216f(a+h) + 27f(a+2h) + 272f\left(\frac{a+b}{2}\right)\right.$$
$$\left. +27f(b-2h) + 216f(b-h) + 41f(b)\right]$$

$$E_6(f) = -\frac{9h^9}{1400}f^{(8)}(\xi), \qquad \xi \in]a,b[$$

- $n = 7,\ h = \dfrac{b-a}{7}$:

$$I_7(f) = \frac{b-a}{120960}\left[5257f(a) + 25039f(a+h) + 9261f(a+2h) + 20923f(a+3h)\right.$$
$$\left. +20923f(b-3h) + 9261f(b-2h) + 25039f(b-h) + 5257f(b)\right]$$

$$E_7(f) = -\frac{8183h^9}{518400}f^{(8)}(\xi), \qquad \xi \in]a,b[$$

<u>Fórmulas de Newton-Cotes abertas de ordem n:</u>

$$I(f) = \int_a^b f(x)\,dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0, 1, \ldots, n$$

$$x_{j,n} = a + (j+1)h, \quad j = 0, 1, \ldots, n, \qquad h = \frac{b-a}{n+2}$$

$$w_{j,n} = I(l_{j,n}) = \frac{h(-1)^{n-j}}{j!(n-j)!}\int_{-1}^{n+1}\prod_{i=0, i\neq j}^{n}(t-i)\,dt, \qquad w_{j,n} = w_{n-j,n}$$

$$E_n(f) = I(f) - I_n(f) = C_n h^{n+1+\nu_n} f^{(n+\nu_n)}(\xi)$$

$$C_n = \frac{1}{(n+\nu_n)!}\int_{-1}^{n+1} t^{\nu_n - 1}\prod_{i=0}^{n}(t-i)\,dt, \qquad \nu_n = 1 + \frac{1}{2}\left[1 + (-1)^n\right], \qquad \xi \in]a,b[$$

- $n = 0,\ h = \dfrac{b-a}{2}$ (Regra do ponto médio):

$$I_0(f) = (b-a)\,f\left(\frac{a+b}{2}\right), \qquad E_0(f) = \frac{h^3}{3}f''(\xi), \quad \xi \in]a,b[$$

- $n = 1,\ h = \dfrac{b-a}{3}$:

$$I_1(f) = \frac{b-a}{2}\left[f(a+h) + f(b-h)\right], \qquad E_1(f) = \frac{3h^3}{4}f''(\xi), \quad \xi \in]a,b[$$

- $n = 2, \ h = \dfrac{b-a}{4}$:

$$I_2(f) = \frac{b-a}{3}\left[2f(a+h) - f\left(\frac{a+b}{2}\right) + 2f(b-h)\right]$$

$$E_2(f) = \frac{14h^5}{45} f^{(4)}(\xi), \quad \xi \in]a,b[$$

<u>Fórmulas de Newton-Cotes fechadas compostas:</u>

$$x_j = a + jh_M, \quad j = 0,1,\ldots,M, \qquad h_M = \frac{b-a}{M}, \qquad f_j := f(x_j)$$

- $n = 1$:

$$I_1^{(M)}(f) = \frac{h_M}{2}\left[f_0 + f_M + 2\sum_{j=1}^{M-1} f_j\right]$$

$$E_1^{(M)}(f) = -\frac{b-a}{12}\, h_M^2 f''(\xi), \qquad \xi \in]a,b[$$

- $n = 2$ (M par):

$$I_2^{(M)}(f) = \frac{h_M}{3}\left[f_0 + f_M + 4\sum_{j=1}^{M/2} f_{2j-1} + 2\sum_{j=1}^{M/2-1} f_{2j}\right]$$

$$E_2^{(M)}(f) = -\frac{b-a}{180}\, h_M^4 f^{(4)}(\xi), \qquad \xi \in]a,b[$$

- $n = 3$ (M múltiplo de 3):

$$I_3^{(M)}(f) = \frac{3h_M}{8}\left[f_0 + f_M + 2\sum_{j=1}^{M/3-1} f_{3j} + 3\sum_{j=1}^{M/3}(f_{3j-1} + f_{3j-2})\right]$$

$$E_3^{(M)}(f) = -\frac{b-a}{80}\, h_M^4 f^{(4)}(\xi), \qquad \xi \in]a,b[$$

- $n = 4$ (M múltiplo de 4):

$$I_4^{(M)}(f) = \frac{4h_M}{90}\left[7\,(f_0 + f_M) + 14\sum_{j=1}^{M/4-1} f_{4j} + 32\sum_{j=1}^{M/4}(f_{4j-1} + f_{4j-3}) + 12\sum_{j=1}^{M/4} f_{4j-2}\right]$$

$$E_4^{(M)}(f) = -\frac{2(b-a)}{945}\, h_M^6 f^{(6)}(\xi), \qquad \xi \in]a,b[$$

- $n = 5$ (M múltiplo de 5):

$$I_5^{(M)}(f) = \frac{5h_M}{288}\left[19\,(f_0 + f_M) + 38\sum_{j=1}^{M/5-1} f_{5j} \right.$$
$$\left. +75\sum_{j=1}^{M/5}(f_{5j-1} + f_{5j-4}) + 50\sum_{j=1}^{M/5}(f_{5j-2} + f_{5j-3})\right]$$

$$E_5^{(M)}(f) = -\frac{55(b-a)}{12096}\, h_M^6 f^{(6)}(\xi), \qquad \xi \in]a,b[$$

- $n = 6$ (M múltiplo de 6):

$$I_6^{(M)}(f) = \frac{h_M}{140}\Bigg[41\,(f_0 + f_M) + 82\sum_{j=1}^{M/6-1} f_{6j} + 216\sum_{j=1}^{M/6}(f_{6j-1} + f_{6j-5}) + 27\sum_{j=1}^{M/6}(f_{6j-2} + f_{6j-4}) + 272\sum_{j=1}^{M/6} f_{6j-3}\Bigg]$$

$$E_6^{(M)}(f) = -\frac{3(b-a)}{2800}\, h_M^8 f^{(8)}(\xi), \qquad \xi \in]a,b[$$

- $n = 7$ (M múltiplo de 7):

$$I_7^{(M)}(f) = \frac{h_M}{17280}\Bigg[5257\,(f_0 + f_M) + 10514\sum_{j=1}^{M/7-1} f_{7j} + 25039\sum_{j=1}^{M/7}(f_{7j-1} + f_{7j-6}) + 9261\sum_{j=1}^{M/7}(f_{7j-2} + f_{7j-5}) + 20923\sum_{j=1}^{M/7}(f_{7j-3} + f_{7j-4})\Bigg]$$

$$E_7^{(M)}(f) = -\frac{1169(b-a)}{518400}\, h_M^8 f^{(8)}(\xi), \qquad \xi \in]a,b[$$

Fórmulas de Newton-Cotes abertas compostas:

$$x_j = a + jh_M, \quad j = 0, 1, \ldots, M, \qquad h_M = \frac{b-a}{M}, \qquad f_j := f(x_j)$$

- $n = 0$ (M par):

$$I_0^{(M)}(f) = 2h_M \sum_{j=1}^{M/2} f_{2j-1}, \qquad E_0^{(M)}(f) = \frac{(b-a)h_M^2}{6} f''(\xi), \quad \xi \in]a,b[$$

- $n = 1$ (M múltiplo de 3):

$$I_1^{(M)}(f) = \frac{3h_M}{2}\sum_{j=1}^{M/3}(f_{3j-2} + f_{3j-1})$$

$$E_1^{(M)}(f) = \frac{b-a}{4}\, h_M^2 f''(\xi), \qquad \xi \in]a,b[$$

- $n = 2$ (M múltiplo de 4):

$$I_2^{(M)}(f) = \frac{4h_M}{3}\sum_{j=1}^{M/4}(2f_{4j-3} - f_{4j-2} + 2f_{4j-1})$$

$$E_2^{(M)}(f) = \frac{7(b-a)}{90}\, h_M^4 f^{(4)}(\xi), \qquad \xi \in]a,b[$$

Fórmulas de Gauss:

$$I(f) = \int_a^b w(x) f(x)\,dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0, 1, \ldots, 2n+1$$

$x_{j,n}$, $j = 0, 1, \ldots, n$: zeros do polinómio Φ_{n+1} de grau $n+1$ pertencente ao sistema $\{\Phi_0, \Phi_1, \ldots\}$ de polinómios mónicos ortogonais com respeito ao produto interno $\langle f, g\rangle = I(fg)$.

$$w_{j,n} = I(l_{j,n}) = I(l_{j,n}^2) = -\frac{\langle \Phi_{n+1}, \Phi_{n+1}\rangle}{\Phi'_{n+1}(x_{j,n})\Phi_{n+2}(x_{j,n})}, \qquad j = 0, 1, \ldots, n$$

$$E_n(f) = \frac{\langle \Phi_{n+1}, \Phi_{n+1}\rangle}{(2n+2)!} f^{(2n+2)}(\xi), \qquad \xi \in]a,b[$$

- Fórmulas de Gauss-Legendre ($[a,b] = [-1,1]$, $w(x) \equiv 1$):

$x_{j,n}$, $j = 0, 1, \ldots, n$: zeros do polinómio de Legendre P_{n+1}

$$w_{j,n} = -\frac{2}{(n+2)P'_{n+1}(x_{j,n})P_{n+2}(x_{j,n})}, \qquad j = 0, 1, \ldots, n$$

$$E_n(f) = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3}\, f^{(2n+2)}(\xi), \qquad \xi \in]a,b[$$

∘• $I_0(f) = 2f(0)$

∘• $I_1(f) = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$

∘• $I_2(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

∘• $I_3(f) = w_{0,3} f(x_{0,3}) + w_{1,3} f(x_{1,3}) + w_{2,3} f(x_{2,3}) + w_{3,3} f(x_{3,3})$

$$x_{0,3} = -\sqrt{\frac{1}{7}\left(3 + 2\sqrt{\frac{6}{5}}\right)} = -x_{3,3}, \quad x_{1,3} = -\sqrt{\frac{1}{7}\left(3 - 2\sqrt{\frac{6}{5}}\right)} = -x_{2,3}$$

$$w_{0,3} = \frac{1}{6}\left(3 - \sqrt{\frac{5}{6}}\right) = w_{3,3}, \quad w_{1,3} = \frac{1}{6}\left(3 + \sqrt{\frac{5}{6}}\right) = w_{2,3}$$

- Fórmulas de Gauss-Chebyshev $\left([a,b] = [-1,1],\ w(x) = 1/\sqrt{1-x^2}\right)$:

$$x_{j,n} = -\cos\left(\frac{2j+1}{2n+2}\,\pi\right), \qquad w_{j,n} = \frac{\pi}{n+1}, \qquad j = 0, 1, \ldots, n$$

$$E_n(f) = \frac{\pi}{2^{2n+1}(2n+2)!}\, f^{(2n+2)}(\xi), \qquad \xi \in]a,b[$$

- Fórmulas de Gauss-Legendre compostas:

$$I(f) = \int_a^b f(x)\,dx \quad \approx \quad I_n^{(M)}(f) = \frac{h_M}{2} \sum_{j=0}^{n} w_{j,n} \sum_{m=1}^{M} f\left(x_{j,n}^{(m)}\right)$$

$$x_{j,n}^{(m)} = a + h_M(m-1) + \frac{h_M}{2}\left(x_{j,n}+1\right), \qquad h_M = \frac{b-a}{M}$$

($x_{j,n}$ e $w_{j,n}$ são os nós e os pesos das fórmulas de Gauss-Legendre)

$$E_n^{(M)}(f) = \frac{b-a}{2}\left(\frac{h_M}{2}\right)^{2n+2} \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3}\, f^{(2n+2)}(\xi), \quad \xi \in]a,b[$$

10. Resolução de Equações Diferenciais Ordinárias: Problemas de Valor Inicial

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = Y_0 \end{cases}$$

$$f : D \subset \mathbb{R}^{1+M} \to \mathbb{R}^M, \qquad D \text{ aberto}, \qquad M \in \mathbb{Z}^+$$

$$f \in C(D), \qquad \|f(x,y) - f(x,z)\| \le L\|y-z\|, \quad \forall (x,y),(x,z) \in D$$

$$(x_0, Y_0) \in D$$

<u>Métodos de passo simples:</u>

$$y_{n+1} = y_n + h\,\varphi(x_n, y_n; h)$$

$$x_n = x_0 + nh, \qquad n = 0,1,\ldots,N, \qquad N \in \mathbb{Z}^+$$

$$\varphi : D\times]0,\infty[\to \mathbb{R}^M, \qquad \varphi \in C(D\times]0,\infty[)$$

$$\|\varphi(x,y;h) - \varphi(x,z;h)\| \le K\|y-z\|, \quad \forall (x,y;h),(x,z;h) \in D\times]0,\infty[$$

- Erro de discretização local:

$$\tau(x,y;h) = \frac{1}{h}\left[Z(x+h) - Z(x)\right] - \varphi(x,y;h)$$

$$Z'(t) = f(t, Z(t)), \qquad Z(x) = y$$

- Erro de discretização global:

$$\|Y(x_n) - y_n(h)\| \le e^{K(x_n-x_0)}\|Y(x_0) - y_0(h)\| + \frac{\tau(h)}{K}\left[e^{K(x_n-x_0)} - 1\right]$$

$$\tau(h) = \max_{0\le n\le N} \|\tau(x_n, Y(x_n); h)\|$$

- Método de Euler:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$\tau(x,y;h) = \frac{h}{2}(d_f f)(x,y) + \mathcal{O}(h^2)$$

$$(d_f g)(x,y) = \left(\frac{\partial}{\partial x} + f(x,y)\cdot\nabla_y\right) g(x,y), \qquad \forall g \in C^1(D)$$

- Métodos de Taylor de ordem q:

$$y_{n+1} = y_n + h\sum_{j=0}^{q-1}\frac{h^j}{(j+1)!}\left(d_f^j f\right)(x_n, y_n)$$

$$\tau(x,y;h) = \frac{h^q}{(q+1)!}(d_f^q f)(x,y) + \mathcal{O}(h^{q+1})$$

- Métodos de Runge-Kutta de ordem 2:

$$\varphi(x,y;h) = (1-\gamma)f(x,y) + \gamma f\left(x + \frac{h}{2\gamma}, y + \frac{h}{2\gamma}f(x,y)\right)$$

$$\tau(x,y;h) = \frac{h^2}{6}\left[d_f^2 f(x,y) - 3\,\frac{\partial^2\varphi}{\partial h^2}(x,y;0)\right] + \mathcal{O}(h^3)$$

  - Método de Euler modificado ou do ponto médio ($\gamma = 1$):

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,f(x_n, y_n)\right)$$

  - Método de Runge-Kutta clássico de ordem 2 $\left(\gamma = \frac{3}{4}\right)$:

$$y_{n+1} = y_n + \frac{h}{4}\left[f(x_n, y_n) + 3f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}\,f(x_n, y_n)\right)\right]$$

  - Método de Heun $\left(\gamma = \frac{1}{2}\right)$:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right]$$

- Métodos de Runge-Kutta de ordem 3:

$$\varphi(x,y;h) = \sum_{j=1}^{3}\gamma_j\varphi_j(x,y;h)$$

$$\varphi_1(x,y;h) = f(x,y)$$

$$\varphi_j(x,y;h) = f\left(x + \alpha_j h, y + h\sum_{i=1}^{j-1}\beta_{ji}\varphi_i(x,y;h)\right), \qquad j = 2,3$$

$$\tau(x,y;h) = \frac{h^3}{24}\left[d_f^3 f(x,y) - 4\,\frac{\partial^3\varphi}{\partial h^3}(x,y;0)\right] + \mathcal{O}(h^4)$$

○• Método de Runge-Kutta clássico de ordem 3

$$y_{n+1} = y_n + \frac{h}{6}\left[\varphi_1 + 4\varphi_2 + \varphi_3\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + h, y_n - h\varphi_1 + 2h\varphi_2\right)$$

○• Método de Runge-Kutta-Nystrom de ordem 3

$$y_{n+1} = y_n + \frac{h}{8}\left[2\varphi_1 + 3\varphi_2 + 3\varphi_3\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}\,\varphi_2\right)$$

○• Método de Runge-Kutta-Heun de ordem 3

$$y_{n+1} = y_n + \frac{h}{4}\left[\varphi_1 + 3\varphi_3\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{3}, y_n + \frac{h}{3}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}\,\varphi_2\right)$$

• Métodos de Runge-Kutta de ordem 4:

$$\varphi(x,y;h) = \sum_{j=1}^{5} \gamma_j \varphi_j(x,y;h)$$

$$\varphi_1(x,y;h) = f(x,y)$$

$$\varphi_j(x,y;h) = f\left(x + \alpha_j h, y + h\sum_{i=1}^{j-1} \beta_{ji}\varphi_i(x,y;h)\right), \qquad j = 2,3,4,5$$

$$\tau(x,y;h) = \frac{h^4}{120}\left[d_f^4 f(x,y) - 5\,\frac{\partial^4\varphi}{\partial h^4}(x,y;0)\right] + \mathcal{O}(h^5)$$

○• Método de Runge-Kutta clássico de ordem 4:

$$y_{n+1} = y_n + \frac{h}{6}\left[\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,\varphi_2\right), \qquad \varphi_4 = f(x_n + h, y_n + h\varphi_3)$$

○● Método de Runge-Kutta-Gill de ordem 4:

$$y_{n+1} = y_n + \frac{h}{6}\left[\varphi_1 + (2-\sqrt{2})\varphi_2 + (2+\sqrt{2})\varphi_3 + \varphi_4\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{\sqrt{2}-1}{2}\,h\varphi_1 + \frac{2-\sqrt{2}}{2}\,h\varphi_2\right)$$

$$\varphi_4 = f\left(x_n + h, y_n - \frac{\sqrt{2}}{2}\,h\varphi_2 + \frac{2+\sqrt{2}}{2}\,h\varphi_3\right)$$

○● Método de Runge-Kutta-Merson de ordem 4:

$$y_{n+1} = y_n + \frac{h}{6}\left[\varphi_1 + 4\varphi_4 + \varphi_5\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{3}, y_n + \frac{h}{3}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + \frac{h}{3}, y_n + \frac{h}{6}\,\varphi_1 + \frac{h}{6}\,\varphi_2\right)$$

$$\varphi_4 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{8}\,\varphi_1 + \frac{3h}{8}\,\varphi_3\right)$$

$$\varphi_5 = f\left(x_n + h, y_n + \frac{h}{2}\,\varphi_1 - \frac{3h}{2}\,\varphi_3 + 2h\varphi_4\right)$$

○● Método de Runge-Kutta-Fehlberg de ordem 4:

$$y_{n+1} = y_n + h\left[\frac{25}{216}\varphi_1 + \frac{1408}{2565}\varphi_3 + \frac{2197}{4104}\varphi_4 - \frac{1}{5}\varphi_5\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{4}, y_n + \frac{h}{4}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + \frac{3h}{8}, y_n + \frac{3h}{32}\,\varphi_1 + \frac{9h}{32}\,\varphi_2\right)$$

$$\varphi_4 = f\left(x_n + \frac{12h}{13}, y_n + \frac{1932}{2197}\,h\varphi_1 - \frac{7200}{2197}\,h\varphi_2 + \frac{7296}{2197}\,h\varphi_3\right)$$

$$\varphi_5 = f\left(x_n + h, y_n + \frac{439}{216}\,h\varphi_1 - 8h\varphi_2 + \frac{3680}{513}\,h\varphi_3 - \frac{845}{4104}\,h\varphi_4\right)$$

<u>Métodos multipasso lineares com $p+1$ passos, $p>0$:</u>

$$y_{n+1} = \sum_{k=0}^{p} a_k y_{n-k} + h\sum_{k=-1}^{p} b_k f(x_{n-k}, y_{n-k}), \qquad n \geq p$$

$$|a_p| + |b_p| \neq 0$$

$$x_n = x_0 + nh, \qquad n = 0, 1, \ldots, N, \qquad N \in \mathbb{Z}^+$$

- Erro de discretização local:

$$\tau(x,y;h) = \frac{1}{h}\left[Z(x+h) - \sum_{k=0}^{p} a_k Z(x-kh)\right] - \sum_{k=-1}^{p} b_k f(x-kh, Z(x-kh))$$

$$Z'(t) = f(t, Z(t)), \qquad Z(x) = y$$

- Condições de consistência ($q \geq 1$: ordem de consistência, $f \in C^{q+1}(D)$):

$$\begin{cases} q = 1: & C_0 = 0, \quad C_1 = 0, \quad C_2 \neq 0 \\ q \geq 2: & C_0 = 0, \quad C_1 = C_2 = \cdots = C_q = 0, \quad C_{q+1} \neq 0 \end{cases}$$

$$C_0 = 1 - \sum_{k=0}^{p} a_k, \qquad C_1 = 1 + \sum_{k=0}^{p} k a_k - \sum_{k=-1}^{p} b_k,$$

$$C_j = 1 - \sum_{k=0}^{p} (-k)^j a_k - j \sum_{k=-1}^{p} (-k)^{j-1} b_k, \qquad j = 2, 3, \ldots$$

$$\tau(x,y;h) = \frac{h^q}{(q+1)!} C_{q+1} (d_f^q f)(x,y) + \mathcal{O}(h^{q+1})$$

- Condição da raiz:

$$\rho(r) = r^{p+1} - \sum_{k=0}^{p} a_k r^{p-k} = \prod_{j=0}^{p} (r - r_j)$$

(i) $|r_j| \leq 1, \ \ j = 0, 1, \ldots, p;$ (ii) $|r_j| = 1 \Rightarrow \rho'(r_j) \neq 0$

- Métodos de Adams-Bashforth ($f_m := f(x_m, y_m)$):

$$y_{n+1} = y_n + h \sum_{k=0}^{p} b_k f_{n-k}, \qquad n \geq p$$

$$C_1 = C_2 = \cdots = C_q = 0, \qquad q = p+1$$

$$C_j = 1 - j \sum_{k=0}^{p} (-k)^{j-1} b_k, \qquad j = 1, 2, \ldots, q$$

◦ • $p = 1, \ \ q = 2$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{2}\left[3f_n - f_{n-1}\right] \\ \tau(x,y;h) = \dfrac{5h^2}{12} (d_f^2 f)(x,y) + \mathcal{O}(h^3) \end{cases}$

◦ • $p = 2, \ \ q = 3$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{12}\left[23f_n - 16f_{n-1} + 5f_{n-2}\right] \\ \tau(x,y;h) = \dfrac{3h^3}{8} (d_f^3 f)(x,y) + \mathcal{O}(h^4) \end{cases}$

◦• $p = 3,\quad q = 4$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{24}\,[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}], \\ \tau(x, y; h) = \dfrac{251 h^4}{720}\,(d_f^4 f)(x, y) + \mathcal{O}(h^5) \end{cases}$

◦• $p = 4,\quad q = 5$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{720}\,[1901 f_n - 2774 f_{n-1} + 2616 f_{n-2} \\ \qquad\qquad -1274 f_{n-3} + 251 f_{n-4}], \\ \tau(x, y; h) = \dfrac{95 h^5}{288}\,(d_f^5 f)(x, y) + \mathcal{O}(h^6) \end{cases}$

• Métodos de Adams-Moulton ($f_m := f(x_m, y_m)$):

$$y_{n+1} = y_n + h \sum_{k=-1}^{p} b_k f_{n-k}, \qquad n \geq p$$

$$C_1 = C_2 = \cdots = C_q = 0, \qquad q = p + 2$$

$$C_j = 1 - j \sum_{k=-1}^{p} (-k)^{j-1} b_k, \qquad j = 1, 2, \ldots, q$$

◦• $p = 0,\quad q = 2$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{2}\,[f_{n+1} + f_n] \\ \tau(x, y; h) = -\dfrac{h^2}{12}\,(d_f^2 f)(x, y) + \mathcal{O}(h^3) \end{cases}$

◦• $p = 1,\quad q = 3$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{12}\,[5 f_{n+1} + 8 f_n - f_{n-1}] \\ \tau(x, y; h) = -\dfrac{h^3}{24}\,(d_f^3 f)(x, y) + \mathcal{O}(h^4) \end{cases}$

◦• $p = 2,\quad q = 4$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{24}\,[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}] \\ \tau(x, y; h) = -\dfrac{19 h^4}{720}\,(d_f^4 f)(x, y) + \mathcal{O}(h^5) \end{cases}$

◦• $p = 3,\quad q = 5$: $\begin{cases} y_{n+1} = y_n + \dfrac{h}{720}\,[251 f_{n+1} + 646 f_n - 264 f_{n-1} \\ \qquad\qquad +106 f_{n-2} - 19 f_{n-3}] \\ \tau(x, y; h) = -\dfrac{3 h^5}{160}\,(d_f^5 f)(x, y) + \mathcal{O}(h^6) \end{cases}$