INSTITUTO SUPERIOR TÉCNICO

Mestrado em Engenharia Electrotécnica e de Computadores

Ano Lectivo: 2007/2008 Semestre: 2º

MATEMÁTICA COMPUTACIONAL

Formulário – I

1. Representação de Números e Teoria de Erros

Erro, erro absoluto, erro relativo ($\tilde{x} \approx x$):

$$\text{(i)} \quad x \in \mathbb{R}: \quad e_{\tilde{x}} = x - \tilde{x}, \quad |e_{\tilde{x}}|, \quad \delta_{\tilde{x}} = \frac{e_{\tilde{x}}}{x}, \quad |\delta_{\tilde{x}}| \quad (x \neq 0)$$

$$\text{(ii)} \quad x \in \mathbb{R}^n: \quad e_{\tilde{x}} = x - \tilde{x}, \quad \|e_{\tilde{x}}\|, \quad \delta_{\tilde{x}} = \frac{e_{\tilde{x}}}{\|x\|}, \quad \|\delta_{\tilde{x}}\| \quad (x \neq 0)$$

Representação de números reais (notação científica):

$$x = \sigma\, m\, \beta^t \ \in \mathbb{R} \setminus \{0\}$$

$$\text{(base) } \beta \in \mathbb{N} \setminus \{1\}, \qquad \text{(sinal) } \sigma \in \{+, -\}, \qquad \text{(expoente) } t \in \mathbb{Z}$$

$$\text{(mantissa) } m = (0.a_1 a_2 \ldots)_\beta \in \ [\beta^{-1}, 1[, \qquad a_i \in \{0, 1, \ldots, \beta - 1\}, \qquad a_1 \neq 0$$

Sistema de ponto flutuante:

$$\mathrm{FP}(\beta, n, t^-, t^+) = \{x \in \mathbb{Q}: \ x = \sigma\, m\, \beta^t\} \cup \{0\}$$

$$\beta \in \mathbb{N} \setminus \{1\}, \qquad \sigma \in \{+, -\}, \qquad t^- \leq t \leq t^+, \quad t, t^-, t^+ \in \mathbb{Z}$$

$$m = (0.a_1 a_2 \ldots a_n)_\beta \ \in [\beta^{-1}, 1 - \beta^{-n}], \qquad a_i \in \{0, 1, \ldots, \beta - 1\}, \qquad a_1 \neq 0$$

Arredondamentos:

$$x = \sigma(0.a_1 a_2 \ldots a_n a_{n+1} \ldots)_\beta \times \beta^t \ \in \mathbb{R}, \qquad \mathrm{fl}(x) \in \mathrm{FP}(\beta, n, t^-, t^+)$$

(i) arredondamento por corte:

$$\mathrm{fl}_c(x) = \sigma(0.a_1 a_2 \ldots a_n)_\beta \times \beta^t$$

(ii) arredondamento simétrico (β par):

$$\mathrm{fl}_s(x) = \begin{cases} \sigma(0.a_1 a_2 \ldots a_n)_\beta \times \beta^t, & 0 \leq a_{n+1} < \dfrac{\beta}{2} \\[2ex] \sigma\,[(0.a_1 a_2 \ldots a_n)_\beta + \beta^{-n}] \times \beta^t, & \dfrac{\beta}{2} \leq a_{n+1} < \beta \end{cases}$$

Erros de arredondamento ($x = \sigma m \beta^t \in \mathbb{R}, \quad \tilde{x} = \mathrm{fl}(x) \in \mathrm{FP}(\beta, n, t^-, t^+)$):

(i) arredondamento por corte:

$$|e_{\tilde{x}}| \leq \beta^{t-n}, \qquad |\delta_{\tilde{x}}| \leq \beta^{1-n} =: u_c$$

(ii) arredondamento simétrico:

$$|e_{\tilde{x}}| \leq \frac{1}{2}\,\beta^{t-n}, \qquad |\delta_{\tilde{x}}| \leq \frac{1}{2}\,\beta^{1-n} =: u_s$$

(u_c, u_s: unidade de arredondamento do sistema $\mathrm{FP}(\beta, n, t^-, t^+)$)

Operações aritméticas num sistema de ponto flutuante ($x, y \in \mathbb{R}; \quad \circ = +, -, \times, \div$):

$$x \boxed{\circ} y = \mathrm{fl}(\mathrm{fl}(x) \circ \mathrm{fl}(y))$$

Algarismo significativo:

$$x = \sigma m 10^t \in \mathbb{R}, \qquad \tilde{x} = \sigma(0.a_1 a_2 \ldots a_n)_{10} \times 10^t \in \mathrm{FP}(10, n, t^-, t^+),$$

$$a_i \quad \text{é algarismo significativo de } \tilde{x} \text{ se} \quad |e_{\tilde{x}}| \leq \frac{1}{2}\,10^{t-i}$$

Propagação de erros $\Big(x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \;\; \phi : \mathbb{R}^n \to \mathbb{R}, \;\; \tilde{x} \approx x, \;\; \tilde{\phi} = \mathrm{fl} \circ \phi\Big)$:

$$e_{\phi(\tilde{x})} = \phi(x) - \phi(\tilde{x}) \quad \approx \quad e^L_{\phi(\tilde{x})} = \sum_{k=1}^{n} \frac{\partial \phi}{\partial x_k}(x)\, e_{\tilde{x}_k}$$

$$\delta_{\phi(\tilde{x})} = \frac{e_{\phi(\tilde{x})}}{\phi(x)} \quad \approx \quad \delta^L_{\phi(\tilde{x})} = \sum_{k=1}^{n} p_{\phi, x_k}(x) \delta_{\tilde{x}_k}, \qquad p_{\phi, x_k}(x) = \frac{x_k \frac{\partial \phi}{\partial x_k}(x)}{\phi(x)}$$

$$\delta_{\tilde{\phi}(\tilde{x})} = \frac{\phi(x) - \tilde{\phi}(\tilde{x})}{\phi(x)} \quad \approx \quad \delta^L_{\phi(\tilde{x})} + \delta_{\mathrm{arr}}, \qquad \delta^L_{\phi(\tilde{x})} = \sum_{k=1}^{n} p_{\phi, x_k} \delta_{\tilde{x}_k}, \qquad \delta_{\mathrm{arr}} = \sum_{k=1}^{m} q_k \delta_{\mathrm{arr}_k}$$

2. Métodos Iterativos

Normas vectoriais ($x = (x_1, \ldots, x_n) \in \mathbb{R}^n$):

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \qquad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

(norma da soma) (norma Euclidiana) (norma do máximo)

Coeficiente assimptótico de convergência de ordem $r \geq 1$ de sucessão $\{x_m\}, \; x_m \to x$:

$$K^{[r]}_\infty = \lim_{m \to \infty} \frac{\|x - x_{m+1}\|}{\|x - x_m\|^r}$$

3. Resolução de Equações Não-lineares ($f : \mathbb{R} \to \mathbb{R}$)

Método da bissecção ($f(z) = 0, \ \ f \in C([a,b]), \ \ f(a)f(b) < 0$):

$$x_{m+1} = x_m + \frac{b-a}{2^{m+2}} \operatorname{sgn}[f(a)f(x_m)], \qquad m = 0, 1, \ldots, \qquad x_0 = \frac{a+b}{2}$$

$$|z - x_m| \le \frac{b-a}{2^{m+1}}, \qquad |z - x_{m+1}| \le |x_{m+1} - x_m|$$

Método do ponto fixo ($f(z) = 0 \Leftrightarrow z = g(z)$):

$$\big(\, |g(x) - g(y)| \le L|x - y|, \quad \forall x, y \in I \subset \mathbb{R}, \quad L < 1; \quad g(I) \subset I \, \big)$$

$$x_{m+1} = g(x_m), \qquad m = 0, 1, \ldots$$

$$|z - x_{m+1}| \le L|z - x_m|, \qquad |z - x_m| \le L^m |z - x_0|$$

$$|z - x_m| \le \frac{1}{1-L}|x_{m+1} - x_m|, \qquad |z - x_{m+1}| \le \frac{L}{1-L}|x_{m+1} - x_m|$$

$$|z - x_m| \le \frac{L^m}{1-L}|x_1 - x_0|$$

○• $g'(z) \neq 0, \quad g \in C^1(I), \quad L = \max_{x \in I} |g'(x)| < 1$:

$$z - x_{m+1} = g'(\xi_m)(z - x_m), \qquad \xi_m \in]z; x_m[$$

$$|z - x_{m+1}| \le L|z - x_m|, \qquad |z - x_m| \le L^m |z - x_0|$$

$$\lim_{m \to \infty} \frac{z - x_{m+1}}{z - x_m} = g'(z), \qquad K_\infty^{[1]} = |g'(z)|$$

○• $g^{(r)}(z) = 0, \ r = 1, \ldots, p-1, \ \ g^{(p)}(z) \neq 0, \quad p = 2, 3, \ldots, \quad g \in C^p(I)$

$$z - x_{m+1} = \frac{1}{p!}(-1)^{p+1} g^{(p)}(\xi_m)(z - x_m)^p, \qquad \xi_m \in]z; x_m[$$

$$|z - x_{m+1}| \le K_p |x - x_m|^p, \qquad |z - x_m| \le K_p^{\frac{1}{1-p}} \left(K_p^{\frac{1}{p-1}} |z - x_0| \right)^{p^m}$$

$$K_p = \frac{1}{p!} \max_{x \in I} \left| g^{(p)}(x) \right|$$

$$\lim_{m \to \infty} \frac{z - x_{m+1}}{(z - x_m)^p} = \frac{(-1)^{p+1}}{p!} \, g^{(p)}(z), \qquad K_\infty^{[p]} = \frac{1}{p!} \left| g^{(p)}(z) \right|$$

Método de Newton ($f(z) = 0, \ \ f'(z) \neq 0, \ \ f \in C^2(I)$):

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)}, \qquad m = 0, 1, \ldots$$

$$z - x_{m+1} = -\frac{f''(\xi_m)}{2f'(x_m)} \, (z - x_m)^2 \, , \qquad \xi_m \in]z, x_m[$$

$$|z-x_{m+1}| \le K|z-x_m|^2, \qquad |z-x_m| \le \frac{1}{K}\,(K|z-x_0|)^{2^m}$$

$$K = \frac{\max_{x\in I}|f''(x)|}{2\min_{x\in I}|f'(x)|}$$

$$\lim_{m\to\infty}\frac{z-x_{m+1}}{(z-x_m)^2} = -\frac{f''(z)}{2f'(z)}, \qquad K_\infty^{[2]} = \left|\frac{f''(z)}{2f'(z)}\right|$$

○ • $f^{(r)}(z)=0,\ r=2,\dots,p-1,\ \ f^{(p)}(z)\neq 0, \quad p=3,4,\dots, \quad f\in C^{p+1}(I)$

$$\lim_{m\to\infty}\frac{z-x_{m+1}}{(z-x_m)^p} = \frac{(-1)^{p+1}(p-1)}{p!}\,\frac{f^{(p)}(z)}{f'(z)}, \qquad K_\infty^{[p]} = \frac{p-1}{p!}\left|\frac{f^{(p)}(z)}{f'(z)}\right|$$

<u>Método de Newton μ-modificado</u>:

$$\big(f^{(r)}(z)=0,\ \ r=0,\dots,\mu-1, \quad f^{(\mu)}(z)\neq 0, \quad \mu=2,3,\dots, \quad f\in C^{\mu}(I)\big)$$

$$x_{m+1} = x_m - \mu\,\frac{f(x_m)}{f'(x_m)}, \qquad m=0,1,\dots$$

$$\lim_{m\to\infty}\frac{z-x_{m+1}}{(z-x_m)^2} = -\frac{1}{\mu}\,\frac{h'(z)}{h(z)}, \qquad K_\infty^{[2]} = \frac{1}{\mu}\left|\frac{h'(z)}{h(z)}\right|$$

onde h é tal que $f(x)=(x-z)^{\mu}h(x), \quad h(z)\neq 0$

<u>Método da secante</u> ($f(z)=0, \ \ f'(z)\neq 0, \ \ f\in C^2(I)$):

$$x_{m+1} = x_m - f(x_m)\,\frac{x_m - x_{m-1}}{f(x_m)-f(x_{m-1})}, \qquad m=1,2,\dots$$

$$z-x_{m+1} = -\frac{f''(\eta_m)}{2f'(\xi_m)}\,(z-x_m)\,(z-x_{m-1}), \qquad \xi_m,\eta_m \in]x_{m-1};z;x_m[$$

$$|z-x_{m+1}| \le K\,|z-x_m|\,|z-x_{m-1}|, \qquad |z-x_m| \le \frac{1}{K}\,\delta^{q_m}$$

$$K = \frac{\max_{x\in I}|f''(x)|}{2\min_{x\in I}|f'(x)|}, \quad \delta=\max\{K|z-x_0|, K|z-x_1|\}, \quad q_m : \text{sucessão de Fibonnaci}$$

$$\lim_{m\to\infty}\frac{|z-x_{m+1}|}{|z-x_m|^r} = \left|\frac{f''(z)}{2f'(z)}\right|^{r-1} =: K_\infty^{[r]}, \qquad r=\frac{\sqrt{5}+1}{2}$$

4. Resolução de Sistemas Lineares ($Ax=b, \ \ A\in \mathbb{M}^n(\mathbb{R}), \ \ b,x\in\mathbb{R}^n$)

<u>Normas matriciais induzidas por normas vectoriais</u> ($A=[a_{ij}]\in\mathbb{M}^n(\mathbb{R})$):

$$\|A\|_p = \sup_{x\in\mathbb{R}^n\setminus\{0\}}\frac{\|Ax\|_p}{\|x\|_p}, \qquad p=1,2,\infty$$

$$\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|, \qquad \|A\|_2 = \sqrt{r_\sigma(A^*A)}, \qquad \|A\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|$$

(norma por colunas) (norma Euclidiana) (norma por linhas)

$$r_\sigma(A) = \max_{\lambda\in\sigma(A)}|\lambda|, \qquad \sigma(A): \text{ espectro de } A$$

Número de condição de uma matriz:

$$\text{cond}_p(A) = \|A\|_p\,\|A^{-1}\|_p, \quad p = 1, 2, \infty, \qquad \text{cond}_*(A) = r_\sigma(A)r_\sigma(A^{-1})$$

Condicionamento de sistemas lineares ($Ax = b,\ \tilde{A}\tilde{x} = \tilde{b}$):

$$\frac{\|x-\tilde{x}\|_p}{\|x\|_p} \le \frac{\text{cond}_p(A)}{1-\frac{\|A-\tilde{A}\|_p}{\|A\|_p}\text{cond}_p(A)}\left(\frac{\|A-\tilde{A}\|_p}{\|A\|_p}+\frac{\|b-\tilde{b}\|_p}{\|b\|_p}\right)$$

$$\frac{\|A-\tilde{A}\|_p}{\|A\|_p}\text{cond}_p(A) = \|A-\tilde{A}\|_p\,\|A^{-1}\|_p < 1, \qquad p = 1, 2, \infty$$

Métodos iterativos:

$$Mx^{(k+1)} = -Nx^{(k)} + b, \qquad k = 0, 1, \ldots$$
$$M + N = A = L + D + U$$

$$x^{(k+1)} = Cx^{(k)} + w, \qquad k = 0, 1, \ldots$$
$$C = -M^{-1}N = I - M^{-1}A, \qquad w = M^{-1}b$$

$$\|x - x^{(k+1)}\| \le c\|x - x^{(k)}\|, \qquad \|x - x^{(k)}\| \le c^k\|x - x^{(0)}\|$$
$$\|x - x^{(k)}\| \le \frac{1}{1-c}\|x^{(k+1)} - x^{(k)}\|, \qquad \|x - x^{(k+1)}\| \le \frac{c}{1-c}\|x^{(k+1)} - x^{(k)}\|$$
$$\|x - x^{(k)}\| \le \frac{c^k}{1-c}\|x^{(1)} - x^{(0)}\|, \qquad (c = \|C\| < 1)$$

- Método de Jacobi ($M = D$):

$$x^{(k+1)} = D^{-1}\left[b - (L+U)x^{(k)}\right], \qquad k = 0, 1\ldots$$

- Método de Gauss-Seidel ($M = D + L$):

$$x^{(k+1)} = D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right), \qquad k = 0, 1\ldots$$

- Método de Jacobi modificado $\left(M = \dfrac{D}{\omega},\ \omega \in \mathbb{R}\setminus\{0\}\right)$:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega D^{-1}\left[b - (L+U)x^{(k)}\right], \qquad k = 0, 1, \ldots$$

- Método de Gauss-Seidel modificado ou SOR $\left(M = \dfrac{D}{\omega} + L,\ \ \omega \in \mathbb{R} \setminus \{0\}\right)$:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right), \qquad k = 0, 1, \ldots$$

5. Resolução de Sistemas Não-lineares ($f : \mathbb{R}^n \to \mathbb{R}^n$)

Método do ponto fixo ($f(z) = 0 \Leftrightarrow z = g(z)$):

$$\left(\ \|g(x) - g(y)\| \leq L\|x - y\|,\ \ \forall x, y \in D \subset \mathbb{R}^n, \quad L < 1; \quad g(D) \subset D\ \right)$$

$$\left(\ g \in C^1(D), \quad L = \sup_{x \in D} \|J_g(x)\|\ \right)$$

$$x^{(m+1)} = g(x^{(m)}), \qquad m = 0, 1, \ldots$$

$$\|z - x^{(m+1)}\| \leq L\|z - x^{(m)}\|, \qquad \|z - x^{(m)}\| \leq L^m\|z - x^{(0)}\|$$

$$\|z - x^{(m)}\| \leq \frac{1}{1-L}\|x^{(m+1)} - x^{(m)}\|, \qquad \|z - x^{(m+1)}\| \leq \frac{L}{1-L}\|x^{(m+1)} - x^{(m)}\|$$

$$\|z - x^{(m)}\| \leq \frac{L^m}{1-L}\|x^{(1)} - x^{(0)}\|$$

Método de Newton generalizado ($f(z) = 0, \ \ f \in C^2(D), \ \ \det[J_f(z)] \neq 0$):

$$\begin{cases} x^{(m+1)} = x^{(m)} + \Delta x^{(m)}, \\ J_f(x^{(m)})\Delta x^{(m)} = -f(x^{(m)}), \end{cases} \qquad m = 0, 1, \ldots$$

$$\|z - x^{(m+1)}\| \leq K\|z - x^{(m)}\|^2, \qquad \|z - x^{(m)}\| \leq \frac{1}{K}\left(K\|z - x^{(0)}\|\right)^{2^m}$$

$$K = \frac{M_2}{2M_1} \begin{cases} \dfrac{1}{M_1} = \sup\limits_{x \in D} \|[J_f(x)]^{-1}\|, \\ M_2 = \max\limits_{1 \leq i \leq n} \sup\limits_{x \in D} \|H_{f_i}(x)\|, \quad H_{f_i} \in L^n, \quad (H_{f_i})_{jk} = \dfrac{\partial^2 f_i}{\partial x_j \partial x_k} \end{cases}$$