INSTITUTO SUPERIOR TÉCNICO
Mestrado em Engenharia Física Tecnológica
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MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 29 de Janeiro de 2008

[1]
(a)[20]

O teorema do ponto fixo de Banach diz-nos que g terá um e um só ponto fixo em D se forem satisfeitas as seguintes condições:

- $g(D) \subset D$ • $g \in C^1(\mathbb{R}^3)$ • $\max_{x \in D} \|J_g(x)\|_\infty < 1$

Verifiquemos pois estas condições:

- $$\begin{cases} g_1(x) \in \left[0, \frac{1}{32}\right] \subset \left[-\frac{1}{4}, \frac{1}{4}\right], & \forall x \in D \\ g_2(x) \in \left[-\frac{1}{4}, -\frac{7}{32}\right] \subset \left[-\frac{1}{4}, \frac{1}{4}\right], & \forall x \in D \\ g_3(x) \in \left[0, \frac{1}{32}\right] \subset \left[-\frac{1}{4}, \frac{1}{4}\right], & \forall x \in D \end{cases} \quad \Rightarrow \quad g(D) \subset D$$

- g_1, g_2, g_3 são polinómios nas variáveis x_1, x_2, x_3, logo são continuamente diferenciáveis nestas variáveis.

- $$J_g(x) = \frac{1}{2} \begin{bmatrix} 0 & x_2 & x_3 \\ x_1 & 0 & x_3 \\ x_1 & x_2 & 0 \end{bmatrix}$$

$$\max_{x \in D} \|J_g(x)\|_\infty = \frac{1}{2} \max_{x \in D} \max(|x_2| + |x_3|, |x_1| + |x_3|, |x_1| + |x_2|) = \frac{1}{4} < 1$$

(b)[20]

$$x^{(0)} = [0 \;\; 0 \;\; 0]^T$$

$$x^{(1)} = g(x^{(0)}) = \frac{1}{4} \, [0 \;\; -1 \;\; 0]^T$$

$$x^{(2)} = g(x^{(1)}) = \frac{1}{4} \left[\frac{1}{16} \;\; -1 \;\; \frac{1}{16}\right]^T$$

$$\|z - x^{(2)}\|_\infty \leq \frac{L}{1-L} \|x^{(2)} - x^{(1)}\|_\infty$$

$$L = \max_{x \in D} \|J_g(x)\|_\infty = \frac{1}{4}$$

$$x^{(2)} - x^{(1)} = \frac{1}{64}\,[1 \ \ 0 \ \ 1]^T\,, \qquad \|x^{(2)} - x^{(1)}\|_\infty = \frac{1}{64}$$

$$\|z - x^{(2)}\|_\infty \le \frac{1}{192}$$

(c)[20]

Método da Newton generalizado:

$$\begin{cases} \tilde{x}^{(1)} = \tilde{x}^{(0)} + \Delta\tilde{x}^{(0)} \\ J_f\left(\tilde{x}^{(0)}\right)\Delta\tilde{x}^{(0)} = -f\left(\tilde{x}^{(0)}\right) \end{cases} \qquad \tilde{x}^{(0)} = [\varepsilon \ \ 0 \ \ \varepsilon]^T$$

$$A_\varepsilon y_\varepsilon = b_\varepsilon : \qquad A_\varepsilon = J_f\left(\tilde{x}^{(0)}\right), \qquad y_\varepsilon = \Delta\tilde{x}^{(0)}, \qquad b_\varepsilon = -f\left(\tilde{x}^{(0)}\right)$$

$$J_f(x) = \begin{bmatrix} -4 & 2x_2 & 2x_3 \\ 2x_1 & -4 & 2x_3 \\ 2x_1 & 2x_2 & -4 \end{bmatrix}$$

$$A_\varepsilon = \begin{bmatrix} -4 & 0 & 2\varepsilon \\ 2\varepsilon & -4 & 2\varepsilon \\ 2\varepsilon & 0 & -4 \end{bmatrix}, \qquad b_\varepsilon = \begin{bmatrix} 4\varepsilon - \varepsilon^2 \\ 1 - 2\varepsilon^2 \\ 4\varepsilon - \varepsilon^2 \end{bmatrix}$$

(d)[20]

O método de Gauss-Seidel convergirá para a solução do sistema $A_\varepsilon y_\varepsilon = b_\varepsilon$ se e só se o raio espectral da matriz iteradora do método for inferior à unidade, $r_\sigma(C_{GS}) < 1$.

$$A_\varepsilon = M_{GS} + N_{GS}, \qquad M_{GS} = \begin{bmatrix} -4 & 0 & 0 \\ 2\varepsilon & -4 & 0 \\ 2\varepsilon & 0 & -4 \end{bmatrix}, \qquad N_{GS} = \begin{bmatrix} 0 & 0 & 2\varepsilon \\ 0 & 0 & 2\varepsilon \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{GS}^{-1} = -\frac{1}{64}\begin{bmatrix} 16 & 0 & 0 \\ 8\varepsilon & 16 & 0 \\ 8\varepsilon & 0 & 16 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ \varepsilon & 2 & 0 \\ \varepsilon & 0 & 2 \end{bmatrix}$$

$$C_{GS} = -M_{GS}^{-1}N_{GS} = \frac{\varepsilon}{4}\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & \varepsilon + 2 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

$$\det(C_{GS} - \lambda I) = \begin{vmatrix} -\lambda & 0 & \frac{\varepsilon}{2} \\ 0 & -\lambda & \frac{\varepsilon}{4}(\varepsilon+2) \\ 0 & 0 & \frac{\varepsilon^2}{4} - \lambda \end{vmatrix} = \lambda^2 \left(\frac{\varepsilon^2}{4} - \lambda \right)$$

Valores próprios de C_{GS} : $0, \ 0, \ \frac{\varepsilon^2}{4}$

$$r_\sigma(C_{GS}) = \frac{\varepsilon^2}{4}$$

O método de Gauss-Seidel convergirá para $\varepsilon \in]-2, 2[$.

(e)[20]

$$\frac{\|y_\varepsilon - y_0\|_1}{\|y_0\|_1} \leq \frac{\text{cond}_1(A_0)}{1 - \frac{\|A_\varepsilon - A_0\|_1}{\|A_0\|_1}\text{cond}_1(A_0)} \left(\frac{\|A_\varepsilon - A_0\|_1}{\|A_0\|_1} + \frac{\|b_\varepsilon - b_0\|_1}{\|b_0\|_1} \right)$$

$$\|A_\varepsilon - A_0\|_1 \|{A_0}^{-1}\|_1 < 1$$

$$A_0 = -4I, \qquad A_0^{-1} = -\frac{1}{4}\, I$$

$$\|A_0\|_1 = 4, \qquad \|A_0^{-1}\|_1 = \frac{1}{4}, \qquad \text{cond}_1(A_0) = \|A_0\|_1 \|A_0^{-1}\|_1 = 1$$

$$b_0 = [0 \ \ 1 \ \ 0]^T, \qquad \|b_0\|_1 = 1$$

$$y_0 = -\frac{1}{4}\, b_0, \qquad \|y_0\|_1 = \frac{1}{4}$$

$$A_\varepsilon - A_0 = \begin{bmatrix} 0 & 0 & 2\varepsilon \\ 2\varepsilon & 0 & 2\varepsilon \\ 2\varepsilon & 0 & 0 \end{bmatrix}, \qquad \|A_\varepsilon - A_0\|_1 = 4\varepsilon$$

$$b_\varepsilon - b_0 = [4\varepsilon - \varepsilon^2 \ \ -2\varepsilon^2 \ \ 4\varepsilon - \varepsilon^2]^T$$

$$\|b_\varepsilon - b_0\|_1 = 8\varepsilon - 2\varepsilon^2 + 2\varepsilon^2 = 8\varepsilon, \quad \forall \varepsilon \in \left[0, \frac{1}{10}\right]$$

$$\|y_\varepsilon - y_0\|_1 \leq \frac{9}{4}\, \frac{\varepsilon}{1-\varepsilon} \leq \frac{1}{4}, \quad \forall \varepsilon \in \left[0, \frac{1}{10}\right]$$

Exame de 29.JAN.2008

[2]

(a)[20]

Fórmula interpoladora de Lagrange:

$$p_3(x) = \sum_{j=0}^{3} f(x_j) l_j(x), \qquad l_j(x) = \prod_{i=0, i\neq j}^{3} \frac{x - x_i}{x_j - x_i}$$

$$p_4(x) = l_1(x) + 5l_2(x) + 14l_3(x)$$

$$l_1(x) = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}\, x(x-2)(x-3)$$

$$l_2(x) = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = -\frac{1}{2}\, x(x-1)(x-3)$$

$$l_3(x) = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{1}{6}\, x(x-1)(x-2)$$

$$\begin{aligned} p_3(x) &= \frac{x}{6}[3(x-2)(x-3) - 15(x-1)(x-3) + 14(x-1)(x-2)] \\ &= \frac{x}{6}\,(2x^2 + 3x + 1) \end{aligned}$$

(b)[15]

$$e_3(x) = f(x) - p_3(x) = \frac{f^{(4)}(\xi)}{4!}\, W_4(x), \qquad x \in [0,3]$$

$$W_4(x) = x(x-1)(x-2)(x-3), \qquad \xi \in \text{int}[0;3;x] \subset [0,3]$$

$$|f(x) - p_3(x)| \leq \frac{e}{24} \max_{x\in[0,3]} |W_4(x)|, \quad \forall x \in [0,3]$$

$$W_4(x) = x^4 - 6x^3 + 11x^2 - 6x$$

$$W_4'(x) = 4x^3 - 18x^2 + 22x - 6 = 4(x - z_1)(x - z_2)(x - z_3)$$

$$z_1 = \frac{3 - \sqrt{5}}{2}, \qquad z_2 = \frac{3}{2}, \qquad z_3 = \frac{3 + \sqrt{5}}{2}$$

$$\max_{x\in[0,3]} |W_4(x)| = \max\{|W_4(z_1)|, |W_4(z_2)|, |W_4(z_3)|\} = \max\left\{1, \frac{9}{16}, 1\right\} = 1$$

$$|f(x) - p_3(x)| \leq \frac{e}{24}, \quad \forall x \in [0,3]$$

Exame de 29.JAN.2008

(c)[15]

Tabela de diferenças divididas:

x_i	$f[x_i]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot]$
x	$f(x)$				
		$(x+1)^2$			
$x+1$	$f(x+1)$		$\dfrac{2x+3}{2}$		
		$(x+2)^2$		$\dfrac{1}{3}$	
$x+2$	$f(x+2)$		$\dfrac{2x+5}{2}$		0
		$(x+3)^2$		$\dfrac{1}{3}$	
$x+3$	$f(x+3)$		$\dfrac{2x+7}{2}$		
		$(x+4)^2$			
$x+4$	$f(x+4)$				

(d)[20]

Melhor aproximação mínimos quadrados discreta:

$$q(x) = a\phi_0(x) + b\phi_1(x) + c\phi_2(x), \qquad q^*(x) = a^*\phi_0(x) + b^*\phi_1(x) + c^*\phi_2(x)$$

$$\phi_0(x) = x(x-1), \qquad \phi_1(x) = (x-1)(x-2), \qquad \phi_2(x) = (x-2)(x-3)$$

$$\sum_{i=0}^{3}[f(x_i) - q^*(x_i)]^2 = \min_{a,b,c} \sum_{i=0}^{3}[f(x_i) - q(x_i)]^2$$

$$\psi \in C(\mathbb{R}) \quad \Rightarrow \quad \bar{\psi} = [\psi(x_0) \;\; \psi(x_1) \;\; \psi(x_2) \;\; \psi(x_3)]^T \in \mathbb{R}^4$$

$$\bar{\phi}_0 = [0 \;\; 0 \;\; 2 \;\; 6]^T, \qquad \bar{\phi}_1 = [2 \;\; 0 \;\; 0 \;\; 2]^T, \qquad \bar{\phi}_2 = [6 \;\; 2 \;\; 0 \;\; 0]^T$$

$$\bar{f} = [0 \;\; 1 \;\; 5 \;\; 14]^T$$

$$\bar{q} = a\bar{\phi}_0 + b\bar{\phi}_1 + c\bar{\phi}_2, \qquad \bar{q}^* = a^*\bar{\phi}_0 + b^*\bar{\phi}_1 + c^*\bar{\phi}_2$$

$$\sum_{i=0}^{3}[\bar{f}[i] - \bar{q}^*[i]]^2 = \min_{a,b,c} \sum_{i=0}^{3}[\bar{f}[i] - \bar{q}[i]]^2$$

$$\begin{bmatrix} \langle\bar{\phi}_0,\bar{\phi}_0\rangle & \langle\bar{\phi}_0,\bar{\phi}_1\rangle & \langle\bar{\phi}_0,\bar{\phi}_2\rangle \\ \langle\bar{\phi}_1,\bar{\phi}_0\rangle & \langle\bar{\phi}_1,\bar{\phi}_1\rangle & \langle\bar{\phi}_1,\bar{\phi}_2\rangle \\ \langle\bar{\phi}_2,\bar{\phi}_0\rangle & \langle\bar{\phi}_2,\bar{\phi}_1\rangle & \langle\bar{\phi}_2,\bar{\phi}_2\rangle \end{bmatrix} \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix} = \begin{bmatrix} \langle\bar{f},\bar{\phi}_0\rangle \\ \langle\bar{f},\bar{\phi}_1\rangle \\ \langle\bar{f},\bar{\phi}_2\rangle \end{bmatrix}$$

$$\langle \bar{\chi}, \bar{\psi} \rangle = \sum_{i=0}^{3} \bar{\chi}[i]\bar{\psi}[i], \qquad \forall \bar{\chi}, \bar{\psi} \in \mathbb{R}^4$$

$$\langle \bar{\phi}_0, \bar{\phi}_0 \rangle = 0 + 0 + 4 + 36 = 40$$

$$\langle \bar{\phi}_0, \bar{\phi}_1 \rangle = 0 + 0 + 0 + 12 = 12$$

$$\langle \bar{\phi}_0, \bar{\phi}_2 \rangle = 0 + 0 + 0 + 0 = 0$$

$$\langle \bar{\phi}_1, \bar{\phi}_0 \rangle = \langle \bar{\phi}_0, \bar{\phi}_1 \rangle = 12$$

$$\langle \bar{\phi}_1, \bar{\phi}_1 \rangle = 4 + 0 + 0 + 4 = 8$$

$$\langle \bar{\phi}_1, \bar{\phi}_2 \rangle = 12 + 0 + 0 + 0 = 12$$

$$\langle \bar{\phi}_2, \bar{\phi}_0 \rangle = \langle \bar{\phi}_0, \bar{\phi}_2 \rangle = 0$$

$$\langle \bar{\phi}_2, \bar{\phi}_1 \rangle = \langle \bar{\phi}_1, \bar{\phi}_2 \rangle = 12$$

$$\langle \bar{\phi}_2, \bar{\phi}_2 \rangle = 36 + 4 + 0 + 0 = 40$$

$$\langle \bar{f}, \bar{\phi}_0 \rangle = 0 + 0 + 10 + 84 = 94$$

$$\langle \bar{f}, \bar{\phi}_1 \rangle = 0 + 0 + 0 + 28 = 28$$

$$\langle \bar{f}, \bar{\phi}_2 \rangle = 0 + 2 + 0 + 0 = 2$$

$$\begin{bmatrix} 40 & 12 & 0 \\ 12 & 8 & 12 \\ 0 & 12 & 40 \end{bmatrix} \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix} = \begin{bmatrix} 94 \\ 28 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 40 & 12 & 0 & \vdots & 94 \\ 12 & 8 & 12 & \vdots & 28 \\ 0 & 12 & 40 & \vdots & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 20 & 6 & 0 & \vdots & 47 \\ 0 & 22 & 60 & \vdots & -1 \\ 0 & 0 & 20 & \vdots & 7 \end{bmatrix}$$

$$\begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 53 \\ -20 \\ 7 \end{bmatrix}$$

$$q^*(x) = \frac{1}{20}\left[53x(x-1) - 20(x-1)(x-2) + 7(x-2)(x-3)\right]$$

Exame de 29.JAN.2008

[3]

$$y'(x) = f(y(x)), \qquad f(y) = \frac{1}{1+y^2}$$

(a)[10]

Método de Euler (passo h):

$$y_{n+1} = y_n + hf(y_n), \qquad n \geq 0$$

$$y_1 = y_0 + hf(y_0) = 1 + \frac{h}{2}$$

$$y_2 = y_1 + hf(y_1) = 1 + \frac{h}{2} + \frac{h}{1+\left(1+\frac{h}{2}\right)^2}$$

(b)[10]

Método de Runge-Kutta clássico de 2ª ordem (passo $2h$):

$$\hat{y}_1 = y_0 + \frac{h}{2}\left[f(y_0) + 3f\left(y_0 + \frac{4h}{3}f(y_0)\right)\right]$$

$$\hat{y}_1 = 1 + \frac{h}{2}\left[\frac{1}{2} + \frac{3}{1+\left(1+\frac{2h}{3}\right)^2}\right]$$

(c)[10]

$$y_{n+1}^{(j+1)} = g\left(y_{n+1}^{(j)}\right), \qquad j = 0, 1, 2, \ldots$$

$$g(u) = y_n + \frac{h}{2}\left[f(y_n) + f(u)\right] = y_n + \frac{h}{2}\left[\frac{1}{1+y_n^2} + \frac{1}{1+u^2}\right]$$

A iteração será convergente se $\max_{u\in\mathbb{R}} |g'(u)| < 1$

$$g'(u) = -h\,\frac{u}{(1+u^2)^2}$$

$$g''(u) = -h\,\frac{1-3u^2}{(1+u^2)^3}$$

$$\max_{u\in\mathbb{R}} |g'(u)| = \left|g'\left(\frac{1}{\sqrt{3}}\right)\right| = \frac{9h}{16\sqrt{3}}$$

A iteração será pois convergente para $h < \dfrac{16\sqrt{3}}{9}$.

Exame de 29.JAN.2008